REMOVAL OF DOMINATED ACTIONS

Instead of reasoning on what agents would do, we take a "backward" approach and think about what they would ~~not~~ do:

DOMINATION

Let's denote with Σ_{-i} the set of all σ_{-i}

Given $\sigma_i, \sigma_i' \in \Sigma$

- σ_i STRICTLY DOMINATES σ_i' IFF

$$\forall \sigma_{-i} \in \Sigma_{-i} \quad u_i(\sigma_i, \sigma_{-i}) > u_i(\sigma_i', \sigma_{-i})$$

- σ_i WEAKLY DOMINATES σ_i' IFF

$$\forall \sigma_{-i} \in \Sigma_{-i} \quad u_i(\sigma_i, \sigma_{-i}) \geq u_i(\sigma_i', \sigma_{-i})$$

$$\exists \sigma_{-i}' \in \Sigma_{-i} \quad u(\sigma_i, \sigma_{-i}') > u_i(\sigma_i', \sigma_{-i}')$$

- σ_i VERY WEAKLY DOMINATES σ_i' IFF

$$\forall \sigma_{-i} \in \Sigma_{-i} \; u_i(\sigma_i, \sigma_{-i}) \geq u_i(\sigma_i', \sigma_{-i})$$

DOMINANT STRATEGY: a strategy that dominates any other strategy (in one of the three flavors defined above)

- Dominant strategies are quite rare in reality
- If they exists, what should be expect from them?

One usually thinks that dominant strategies are also "good" in some sense: if I have a dominant strategy then it's better to play it.

Let's see this example (a very famous game in GT)

- 2 criminals have been arrested
- the police can prove a minor charge but not the major one
- criminal are kept in separated rooms and offered the following bargain:

  - BETRAY THE OTHER (Cooperate, C)
  - STAY SILENT (S)

	S	C
S	-1, -1	-4, 0
C	0, -4	-3, -3

- Let's not already think about what we would do ...

- If you were an external, unbiased observer, what outcomes would you favor? (In pure strategies)

  $PE = \{(S,S); (C,S); (S,C)\}$

- Let's see what happens when reasoning in dominant strategies

	S	C
~~S~~	~~-1, -1~~	~~-4, 0~~
C	0, -4	-3, -3

FOR PLAYER ① C STRICTLY DOMINATES S

FOR PLAYER ② IS THE SAME

→ DOMINANT STRATEGY

the dominant strategy makes us play the *ONLY* non Pareto-efficient outcome !!

Nevertheless, we can define a solution concept over this iterative procedure we just used: ITERATED DOMINANCE (strict, weak, very weak)

→ Iteratively removes from the game dominated <u>actions</u>

(it suffices to consider actions, since a dominated action will never be played also in mixed strategies)

IT IS NOT USEFUL TO REMOVE DOMINATED MIXED STRATEGIES

example:

2 \ 1	d	e	f
a	9, 3	10, 4	2, 2
b	2, 1	1, 2	0, 15
c	5, 6	4, 7	3, 9

Solution: all σ such that $\sigma_1(b) = 0$ and $\sigma_2(d) = 0$

$$SID = \{ \sigma \mid \sigma_1(b) = \sigma_2(d) = 0 \}$$

Does the order of elimination count?

- STRICT DOMINANCE: NO
- WEAK AND VERY WEAK: YES

Let's discuss dominance from an algorithmic stance:

① DECIDE IF A STRATEGY σ_i IS (STRICTLY) DOMINATED BY A PURE STRATEGY a_i

(Notice that σ_i is, in general, mixed. Although it's not very useful to remove dominated mixed strategies, like we saw before, we here state the problem generally)

Simple algorithm: IS σ_i STRICTLY DOMINATED BY SOME a_i?

```
FOR ALL a_i ∈ A_i , a_i ≠ σ_i
    dominated ← TRUE
    FOR ALL a_-i ∈ A_-i
        IF u_i(σ_i, a_-i) ≥ u_i(a_i, a_-i)
            dominated ← FALSE
            break;

RETURN dominated
```

runs in $O(|A|)$
(linear in the size of the game)

IS $\underbrace{\sigma_i}$ DOMINATED BY a_i?

we saw how (in iterated dominance) it is convenient to consider *pure* σ_i
(if an action is dominated, then so it is any σ_i that has that action in the support)

THE ALGORITHM IS COMPARING σ_i AND a_i CONSIDERING ONLY PURE STRATEGIES FOR THE OTHER PLAYERS ($a_{-i} \in A_{-i}$). IS IT SAFE?

(The definition of dominance was stating the other way... considering mixed str.)

Let's formally prove this:

Given a_i, σ_i

HP: $u_i(a_i, a_{-i}) > u_i(\sigma_i, a_{-i}) \quad \forall a_{-i} \in A_{-i}$ (this is the condition that the algorithm checks)

TH: $u_i(a_i, \sigma_{-i}) \leq u_i(\sigma_i, \sigma_{-i})$

If we prove TH the the algorithm is not sound because

- the condition checked by the algorithm holds, so the algorithm will say "a_i DOMINATES STRICTLY σ_i"
- Since TH holds a_i DOES NOT DOMINATE STRICTLY σ_i

[redacted]

from [redacted] TH:

$$\sum_{a_{-i}} \sigma_{-i}(a_{-i})\, u_i(a_i, a_{-i}) \leq \sum_{a_{-i}} \sigma_{-i}(a_{-i})\, u_i(\sigma_i, a_{-i})$$

then $\exists a_{-i} \quad u_i(a_i, a_{-i}) \leq u_i(\sigma_i, a_{-i})$

which contraddicts the Hp.

② DECIDE IF A STRATEGY σ_i IS (STRICTLY) DOMINATED BY A MIXED STRATEGY σ_i'

We cannot use the enumeration here:

we can formulate a LINEAR FEASIBILITY PROGRAM:

Find a σ' such that:

$$\sum_{a_i \in A_i} \sigma_i'(a_i)\, u_i(a_i, a_{-i}) > u_i(\sigma_i, a_{-i}) \qquad \forall a_{-i} \in A_{-i}$$

$$\sigma_i'(a_i) \geq 0 \qquad \forall a_i \in A_i$$

$$\sum_{a_i \in A_i} \sigma_i'(a_i) = 1$$

No objective function, linear constraint.
A Linear Program can be solved in polynomial time.
There's only one problem: IT'S NOT A LINEAR PROGRAM!
why? Short story: because of the strict inequality >

In general a LP is:

$$\max\ w^T x$$
$$\text{s.t.}\quad Ax \leq b$$
$$x \geq 0$$

the good news is that we can still rewrite our problem as an LP with a "rather standard" trick

$$\min \sum_{a_i \in A_i} \sigma_i'(a_i)$$

s.t.

$$\sum_{a_i \in A_i} \sigma_i'(a_i) u_i(a_i, a_{-i}) \geq u_i(\sigma_i, a_{-i}) \quad \forall a_{-i} \in A_{-i}$$

$$\sigma_i'(a_i) \geq 0 \quad \forall a_i \in A_i$$

notice that in this problem: $\sum_{a_i \in A_i} \sigma_i'(a_i) <>= 1$

this problem is always feasible

WHY THIS WORKS?

- It's a minimization problem so at the optimum at least one of the $\geq$ constraints should hold tight for one a_{-i}

a σ' that strictly dominates σ exists IFF OPT < 1
$\Rightarrow$ we can add positive probability and have the $\geq$ constraint to hold as $>$ for every a_{-i}

Complexity:

SIMPLEX METHODS
- exponentially in the worst case
- efficient in practice
  ↳ SMOOTHED P

INTERIOR POINTS METHOD
- Converge in P time
- sometimes outperform simplex

Simplex is still the most popular

③ ITERATIVE DOMINANCE

n = number of players

m = number of actions

need to solve $O(nm)$ linear programs

(each program has an exponential number of constraints m^n, but also the game is exp. in such dimension)

↓

(there are games with non-exponential representations, Polymatrix games. We will see them if there is time)

Some questions about iterated dominance:

	Strict dominance	Weak, very weak dom.
CAN a_i BE REMOVED WITH ITERATED DOMINANCE?		
CAN A GAME BE REDUCED TO A GAME WITH $A_i' \subseteq A_i$ FOR EACH PLAYER i?	P	NP - Complete
CAN A GAME BE REDUCED TO A GAME WHERE EACH PLAYER i HAS K_i ACTIONS		

The solution concept we saw so far are not very popular/interesting in game theory.

Let's see how they answer the question "How to play a game" or "how to solve a game"

PE : requires external point of view ... in some cases does not say anything (e.g., in zero sum games); "grand coalition" not the individual

DS : very rarely exists

ID : rarely solves a game, brings a trivial message actually (more useful to reduce the size of a game as we will see)

How can we get more insights on how a game will be played?

Let's do an experiment : PRISONERS' DILEMMA

Walter White \ Jesse Pinkman	S	C
S	-1, -1	-4, 0
C	0, -4	-3, -3

Suppose that you are one of these guys

You and the other guy have the same lawyer (Saul Goodman, for example)

Before deciding what to do you have a private hearing with your lawyer

Your lawyer is a TRUSTED GUY (People trust him)

He tells you : "STAY SILENT" $\longrightarrow$ What you will do?

1) I trust my lawyer, so S

2) Wait a minute... the other guy is thinking the same! → C

3) " " → C

⋮

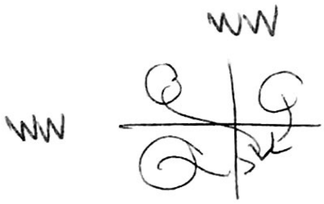
∞ " "

What if the lawyer suggests "cooperate with the police"

⟹ (C, C)

⟹ The lawyer is not really a TRUSTED GUY

(S, S) is somekind of an unsteady state: IF THE STRATEGY OF THE OTHER IS GIVEN THE PLAYER WANTS TO CHANGE HIS AND VICE VERSA!



Lawyer → INTROSPECTION

This way of reasoning about a game based on the concept of steady state is the rationale behind the most popular solution concept in game theory: NASH EQUILIBRIUM (NE)

Let's formalize it by starting from the notion of best response

Best response of player i: Let's suppose that the other agents committed to play σ_{-i}, and that we know this.

→ WHAT'S THE BEST COURSE OF ACTIONS GIVEN SUCH KNOWLEDGE?

$$\sigma_i^* \in \Sigma_i \quad \text{s.t.} \quad u_i(\sigma_i^*, \sigma_{-i}) \geq u_i(\sigma_i, \sigma_{-i})$$

for any other $\sigma_i \in \Sigma_i$

$$\longrightarrow \sigma_i^* = \underset{\sigma_i}{\operatorname{argmax}}\, u_i(\sigma_i, \sigma_{-i})$$

Is the best response unique? NO! The agent can be indifferent between two or more actions:

KEY CONCEPTS: Suppose that σ_i^* ; $S(\sigma_i^*) = \{a_1, a_2\}$

for example:

$$\sigma_i^*(a_1) = 0{,}7 \quad , \quad \sigma_i^*(a_2) = 0{,}3$$

The agent must be indifferent between a_1 and a_2

⟹ a_1 is a best response

a_2 is a best response

ANY MIXTURE BETWEEN a_1 AND a_2 IS A BEST RESPONSE
(e.g., $\sigma_i^*(a_1) = 0{,}5$ $\sigma_i^*(a_2) = 0{,}5$)

NASH EQUILIBRIUM:

$\sigma = (\sigma_1, \sigma_2, \dots, \sigma_M)$ IS A NE IF FOR ALL i σ_i IS A BR TO σ_{-i}

Given the strategy of the others, no agent has a unilateral incentive to change his strategy

Strict Nash:

$\forall i \quad u_i(\sigma_i, \sigma_{-i}) > u_i(\sigma_i', \sigma_{-i}) \qquad \sigma_i' \neq \sigma_i$ Unique best response

Weak Nash:

$\forall i \quad u_i(\sigma_i, \sigma_{-i}) \geq u_i(\sigma_i', \sigma_{-i})$ non unique best response

(MIXED STRATEGY NE ARE WEAK) (PURE STRATEGY NE CAN BE WEAK OR STRICT)

Let's look again to our prisoner's dilemma:

	S	C
S	-1, -1	-4, 0
C	0, -4	$\boxed{-3, -3}$

→ Only Pure NE (strict)

- Are NE Pareto efficient? In general, NO
- Iterated strict dominance preserves all Nash Equilibrium
- Weak and very weak do not, but ....

Let's take another game:

	1, 1	2, 2 (weak)
	3, 2 (strict)	2, 1

two, pure NE

Ok, now let's take this one: MATCHING PENNIES

		j	
		H	T
i	H	$1, -1$	$-1, 1$
	T	$-1, 1$	$1, -1$

- How many pure Nash? ⟶ None... (no strict NE possible)

- BTW, what do you think would be a good strategy to play this game?

⟶ LET'S SUPPOSE THAT AN ORACLE TELLS US THAT PLAYERS WILL RANDOMIZE AT THE EQUILIBRIUM

⟹ Player i must be indifferent between H and T (necessary condition for randomization)

It means that: $u_i(H) = u_i(T)$

⟹ W.l.o.g. Player j plays H with prob. p and T with $(1-p)$

$$u_i(H) = p - (1-p) = 2p - 1$$

$$u_i(T) = -p + (1-p) = -2p + 1$$

$$2p - 1 = -2p + 1$$

$$4p = 2$$

$$\boxed{p = \frac{1}{2}}$$

In order to make player i indifferent between H and T player j must randomize $(1/2, 1/2)$

$\Rightarrow$ BOTH PLAYERS RANDOMIZE $(1/2, 1/2)$

$(1/2, 1/2)$ is not i's unique BR! It's needed to have THE OTHER RANDOMIZE AS WELL

$\downarrow$ This is a NE (weak) in mixed strategies

Sometimes a NE "makes sense" to us, just like in matching pennies

Sometimes it doesn't: in prisoner's dilemma the unique, strict NE is the only outcome which is not Pareto efficient

$\downarrow$

THE REASON IS ACTUALLY THIS: PE DOES NOT IMPLY NE AND NE DOES NOT IMPLY PE!

You can verify that in the battle of the sexes game:

2, 1	0, 0
0, 0	1, 2

each player plays her "favorite show" with $P = 2/3$ and the other with $P = 1/2$

there are also two strict NE in pure strategies

BE CAREFUL! This simple procedure for computing NE in mixed strategies was very simple only because:

- we have two players
- we know the equilibrium supports
- each player has only two actions

Still, we were able to find a mixed NE! Was this by chance? NO

MOST CENTRAL RESULT IN GAME THEORY:

Any game with finite number of players and actions admits at least one Nash Equilibrium (we must admit mixed strategies)
[John Nash, 1950]

Some remarks:

- We saw this game has multiple NE

2, 1	0, 0
0, 0	1, 2

Same holds for the "chicken game"

	T	W
T	-1, -1	2, 1
W	1, 2	0, 0

(NE in 1/2 1/2)

What to do when we have multiple NE?

1) Our analysis is still worth something: we know what the possible rational outcome are

2) But what the players are going to do??

↓

EQUILIBRIUM SELECTION PROBLEM: a mechanism that induces the players to "expect" the same equilibrium

→ many theories ... one example: focal points

WE NEED TO GO TO LUNCH TOGETHER

	main entrance	room 5
main entrance	1, 1	0, 0
room 5	0, 0	1, 1

WE NEED TO MEET AT THE SAME TIME

	12:00 pm	11:53
12:00 pm	1, 1	0, 0
11:53	0, 0	1, 1

the "focal" feature is abstracted away from the game that happens in real-life scenarios.

- There are a number of REFINEMENTS of NE, here some that are worth mentioning

  - ε-NASH EQUILIBRIUM

    a unilateral deviation happens only if a player has a gain in utility which is $\geq \varepsilon$ with $\varepsilon > 0$)

    (in standard Nash $\varepsilon = 0$ (weak))

    (it always exist!)

  - STRONG NASH EQUILIBRIUM

    no group of agents can jointly deviate

    —> IMPLIES PARETO EFFICIENCY

    —> RATHER RARE

    (careful: a NE can be both strong and weak!)

Before tackling the algorithmic aspects of computing NE, let's introduce another solution concept which, as we will see, has a very interesting relationship with NE: MAXMIN and MINMAX

Nash equilibrium describes "rationality" in strategic games in its more natural, basic way.

With maxmin/minmax we study rationality but under two (very related) playing attitudes

- EXTREMELY CAUTIOUS BEHAVIOR ①
- EXTREMELY AGGRESSIVE BEHAVIOUR ②

[WE ASSUME 2-PLAYER GAMES]

① Assumes that a worst case will always happen and then wants to play in a way that mitigates at best the worst case

IT'S A MAXMIN PLAYER: it plays a strategy that maximizes her worst case payoff. → She seeks the best worst case

MAXMIN STRATEGY: $\arg\max_{\sigma_i} \min_{\sigma_{-i}} u_i(\sigma_i, \sigma_{-i})$

MAXMIN VALUE FOR PLAYER i: $\max_{\sigma_i} \min_{\sigma_{-i}} u_i(\sigma_i, \sigma_{-i}) = v_i$

Reasoning in this way amounts to assume that the other player will try to harm/punish us without taking care of her own payoffs. The other player is assumed to be the extremely aggressive player ②, a "foolish Gamer".

Is it reasonable to play in this way? Is it reasonable to be extremely cautious?

- NO: why should the other guy be a "foolish Gamer"?

but

- YES: I can guarantee a utility of v_i without making any assumption about the other player

RATIONALITY
+
NO ASSUMPTIONS
ON THE OPPONENT

$\longrightarrow$

MAXMIN
(equivalent to assume that the other guy is ②)

the maxmin strategy is often called "security strategy" and v_i is often called "security level" of the game.

Let's take the side of player ②

②: is our "foolish lover", wants to guarantee a maximum punishment to the other, she seeks the worst best case for the other

IT'S A MINMAX PLAYER

MINMAX STRATEGY: (2-player games)
$$\arg\min_{\sigma_i} \max_{\sigma_{-i}} u_{-i}(\sigma_i, \sigma_{-i})$$

MINMAX VALUE: (2-player games)
$$\min_{\sigma_i} \max_{\sigma_{-i}} u_{-i}(\sigma_i, \sigma_{-i}) = h_{-i} \quad (\text{for the other!})$$

Maxmin and Minmax values always exists and are unique

- IN ANY 2-PLAYER GAME:

MAXMIN VALUE OF i = MINMAX VALUE OF i

$$v_i = h_i$$

Why?