Sistemi Intelligenti
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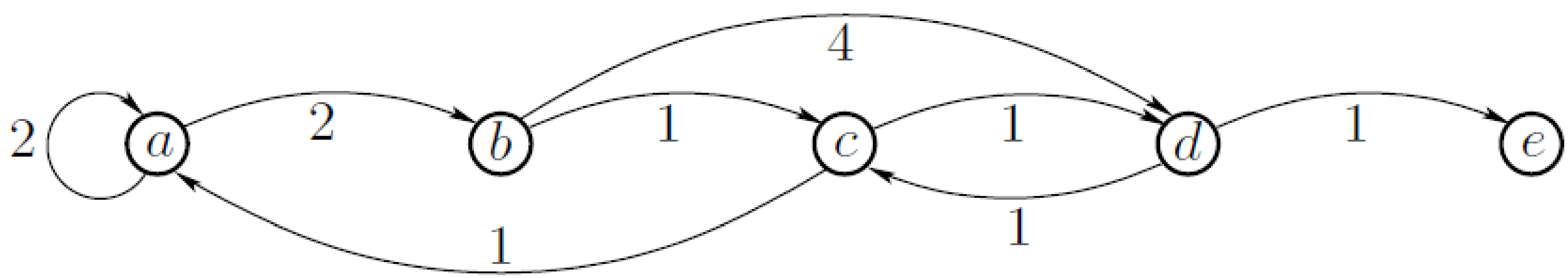
Università degli Studi di Milan

Search algorithms for planning

Nicola Basilico
Dipartimento di Informatica
Via Celoria 18- 20133 Milano (MI)
Ufficio 4008
nicola.basilico@unimi.it
+39 02.503.16289

State-based problem formulation

- (Single agent: the automated problem solver)
- State space defined as a set of **nodes**, each node represents a state; we assume a finite state space
- For each state, we have set of actions that can be undertaken by the agent from that state
- Transition model: given a starting state and an action, indicates an arrival state (here we assume no uncertainties, i.e., deterministic transitions and full observability)
- Action costs: any transition has a cost, which we assume to be greater than a positive constant (reasonable assumption, useful for deriving some properties of the algorithms we discuss)
- Initial state



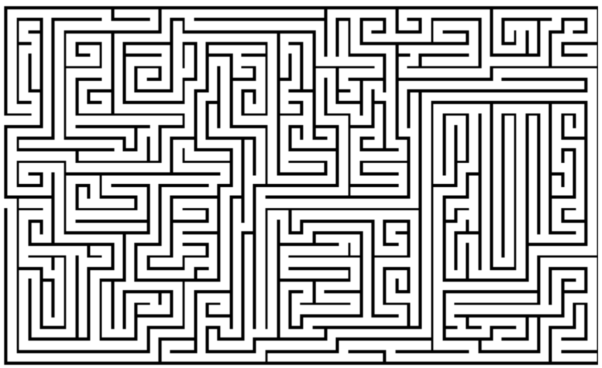
Compact representation: state transition graph G=(V,E)
(We will use "state" and "node" as interchangeable terms)

Formally describing the desired solution

- In the problem formulation we need to formally describe the features of the solution we seek

- Two (three) classes of problems:

feasibility

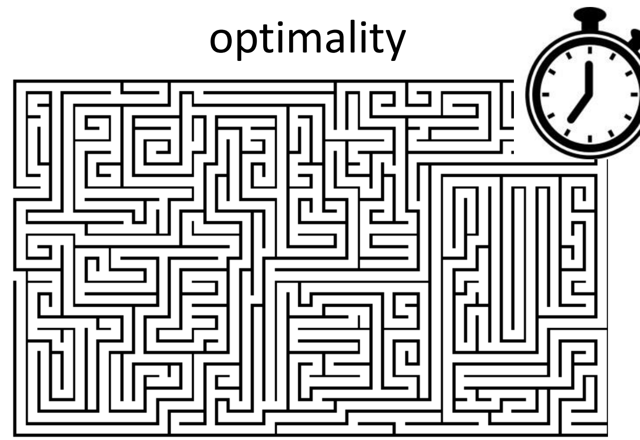


is there a path to an exit?

Set of goal states, find any sequence of actions (path) from the initial state to a goal state

(approximation)

optimality



If at least a path to an exit exists, what is the one with the minimum number of turns?

Set of goal states, find the sequence of actions (path) from the initial state to a goal state that has the minimum cost

Problem example

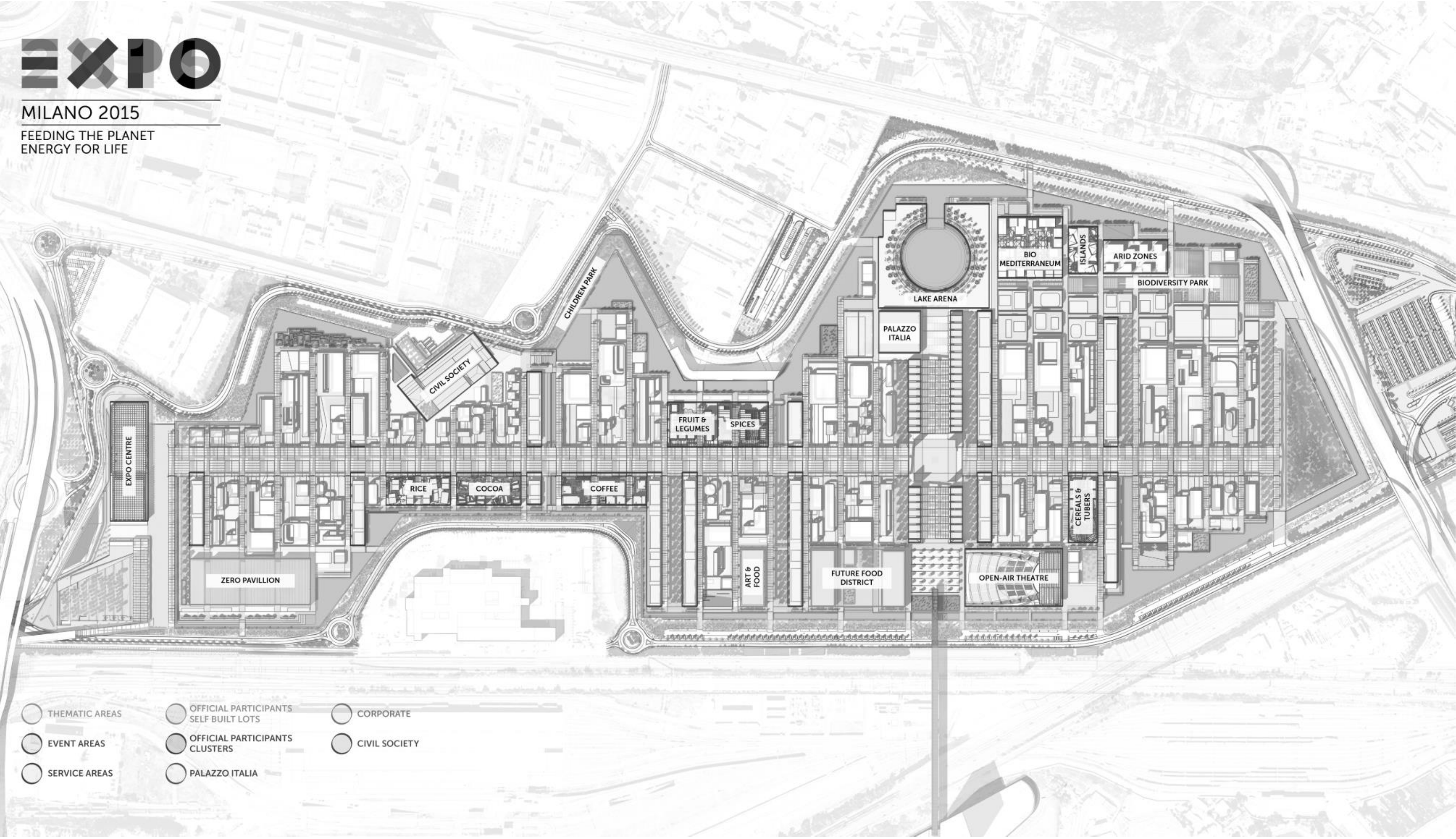
Consider a mobile robot moving on a graph-represented environment:

- **States**: nodes of the graph, they represent physical locations
- **Edges**: represent connections between nearby locations or, equivalently, movement actions
- **Initial state**: some starting location for the robot

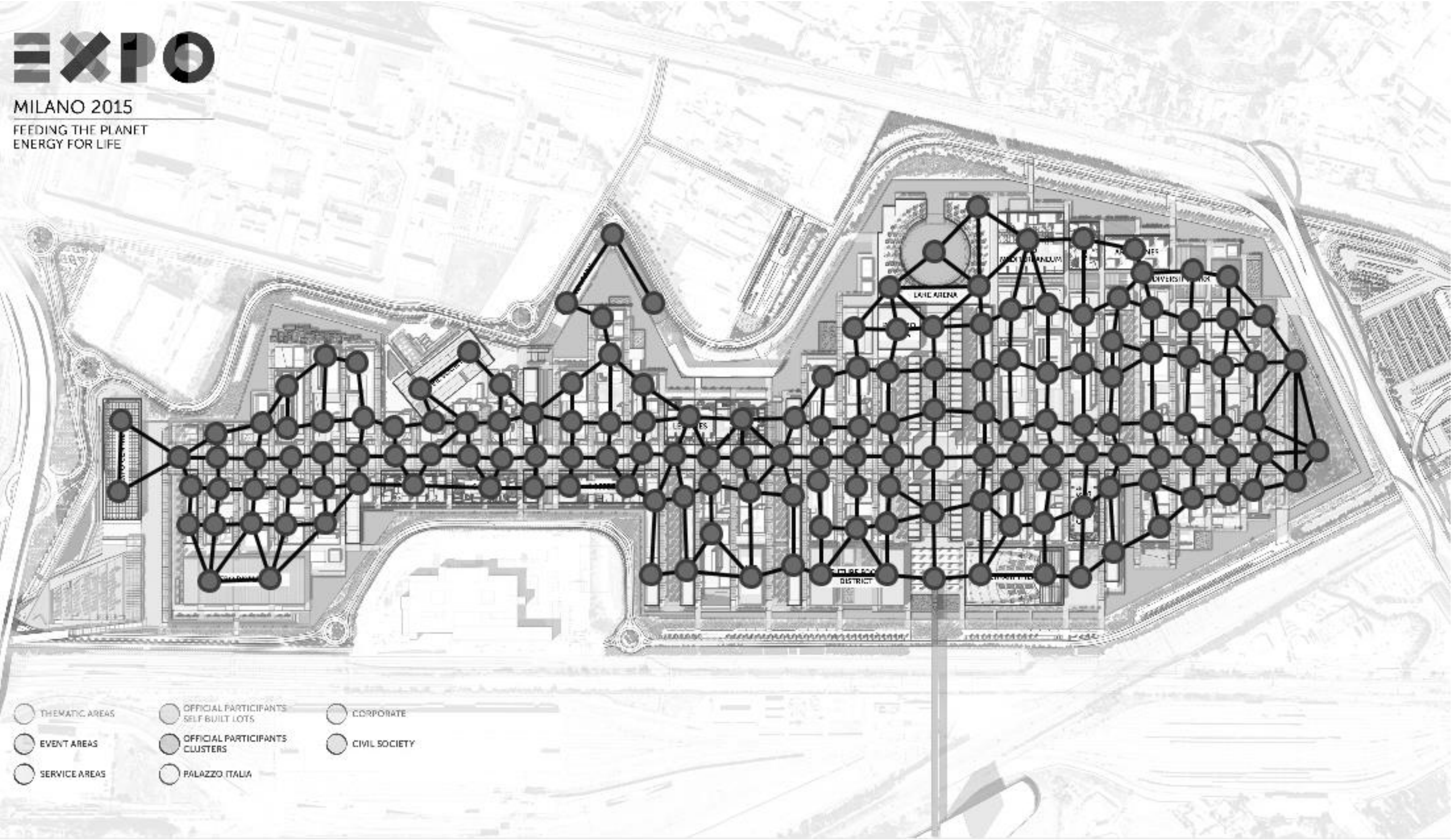
Desired solution:

- **Goal state(s)**: some location(s) to reach, e.g., recharging station, parking depot…
- Find a path to the initial location to a goal one

Problem example



Problem example



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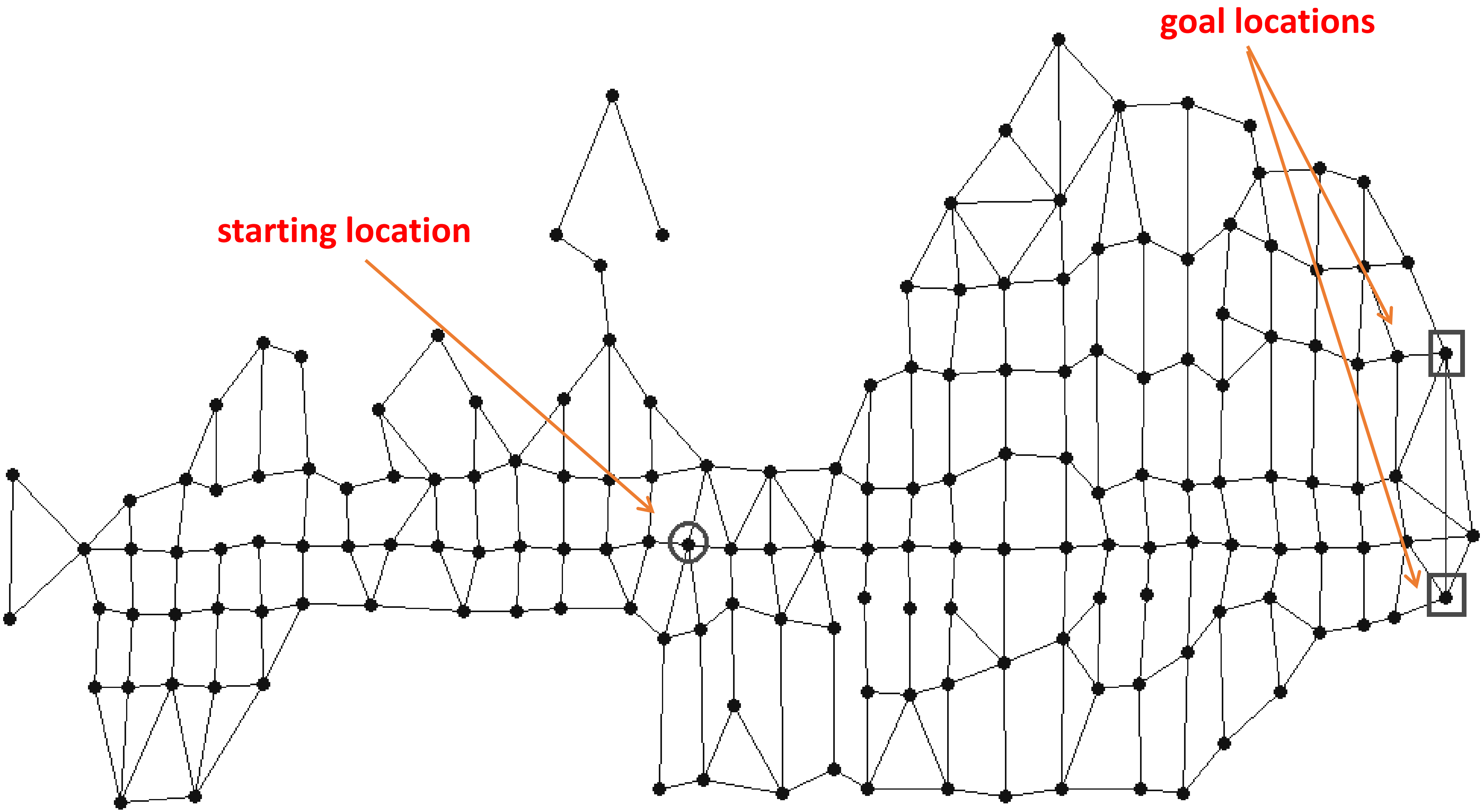
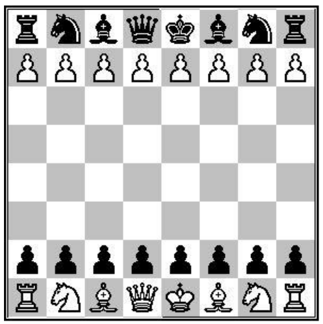


Problem specification

- How to **specify** a planning problem?
- First approach: provide the full state transition graph G (as in the previous example)
- Most of the times this is not an affordable option due to the combinatorial nature of the state space:



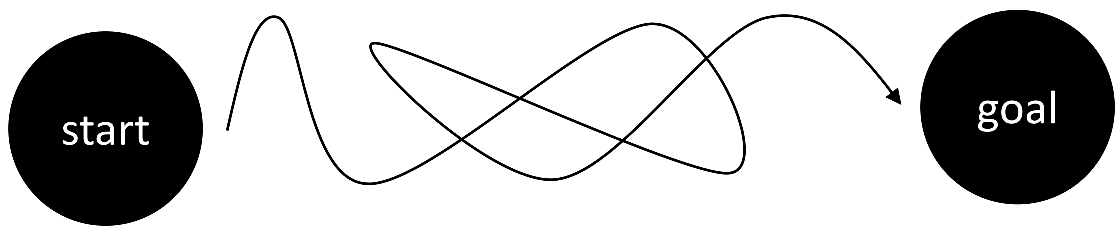
- **Chess board**: approx. 10^{47} states
- We can specify the initial state and the transition function in some compact form (e.g., set of rules to generate next states)
- The planning problem "unfolds" as search progresses

- We need an efficient procedure for *goal checking*

General features of search algorithms

A search algorithm explores the state-transition graph graph G until it discovers the desired solution

- In feasibility: when a goal node is visited the path that led to that node is returned
- In optimality: when a goal node is visited, if any other possible path to that node has higher cost the path that led to that node is returned



It does not suffice to visit a goal node, the algorithm has to reconstruct the path it followed to get there: it must keep a trace of its search

Such a trace can be mapped to a subgraph of G, it is called *search graph*

how to evaluate a (search) algorithm?

- We can evaluate a search algorithm along different dimensions
  - Sound?
  - Complete? (Systematic?)
  - Space complexity?
  - Time complexity?

(The above criteria can actually be used to evaluate a broader class of algorithms)

Soundness

- If the algorithm returns a solution, is it compliant with the desired features specified in the problem formulation?

- Example:
  - In feasibility: *does the returned solution lead to a goal?*
  - In optimality: *does the returned solution lead to a goal with minimum cost?*

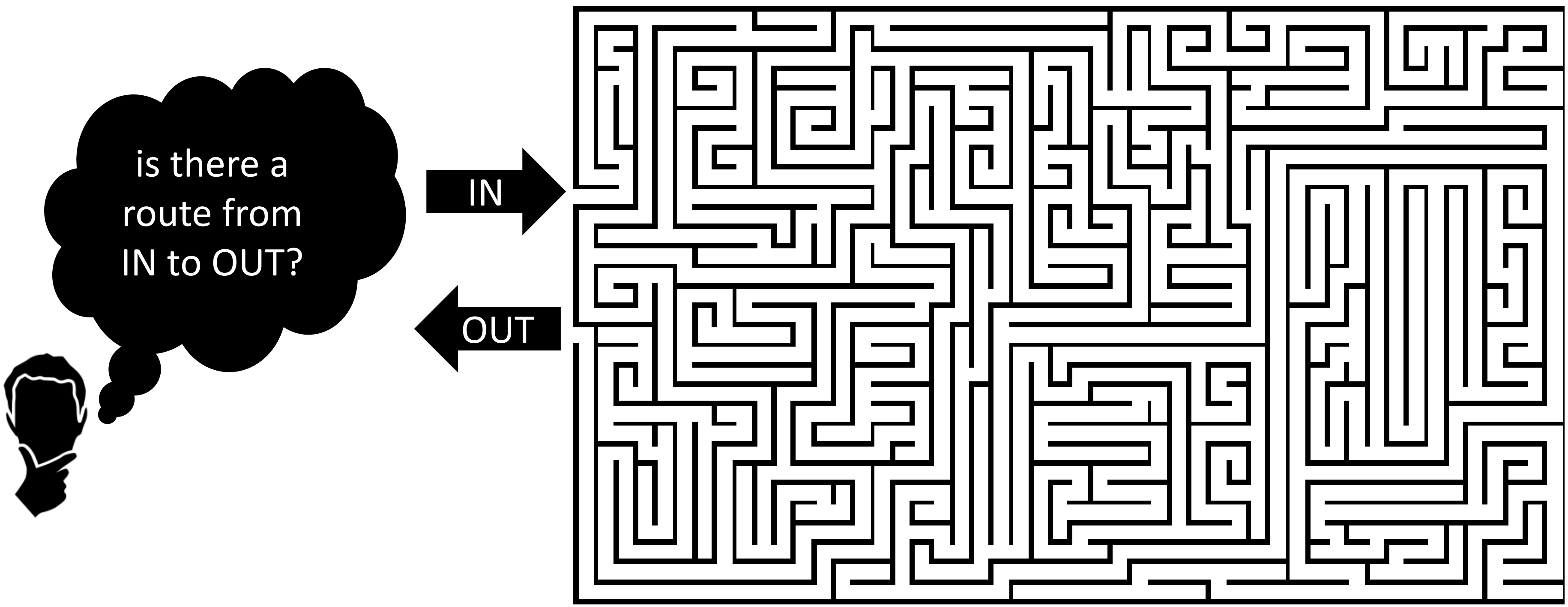
Completeness and the systematic property

- If a solution exists, does the algorithm find it?
- Example:
  - In feasibility: does it always find a path to the goal when it exists?
  - In optimality: does it always find the path to the goal that has minimum cost when at least one exists?

- Typically shown by proving that the search will/will not visit all states if given enough time

- If the state space is infinite, we can ask if the search is systematic:
  - if the answer is “yes” the algorithm must terminate
  - if the answer is “no”, it’s ok if it does not terminate but …
  - … all reachable states must be visited in the limit: as time goes to infinity, all states are visited (this definition is sound under the assumption of countable state space)

Visual example



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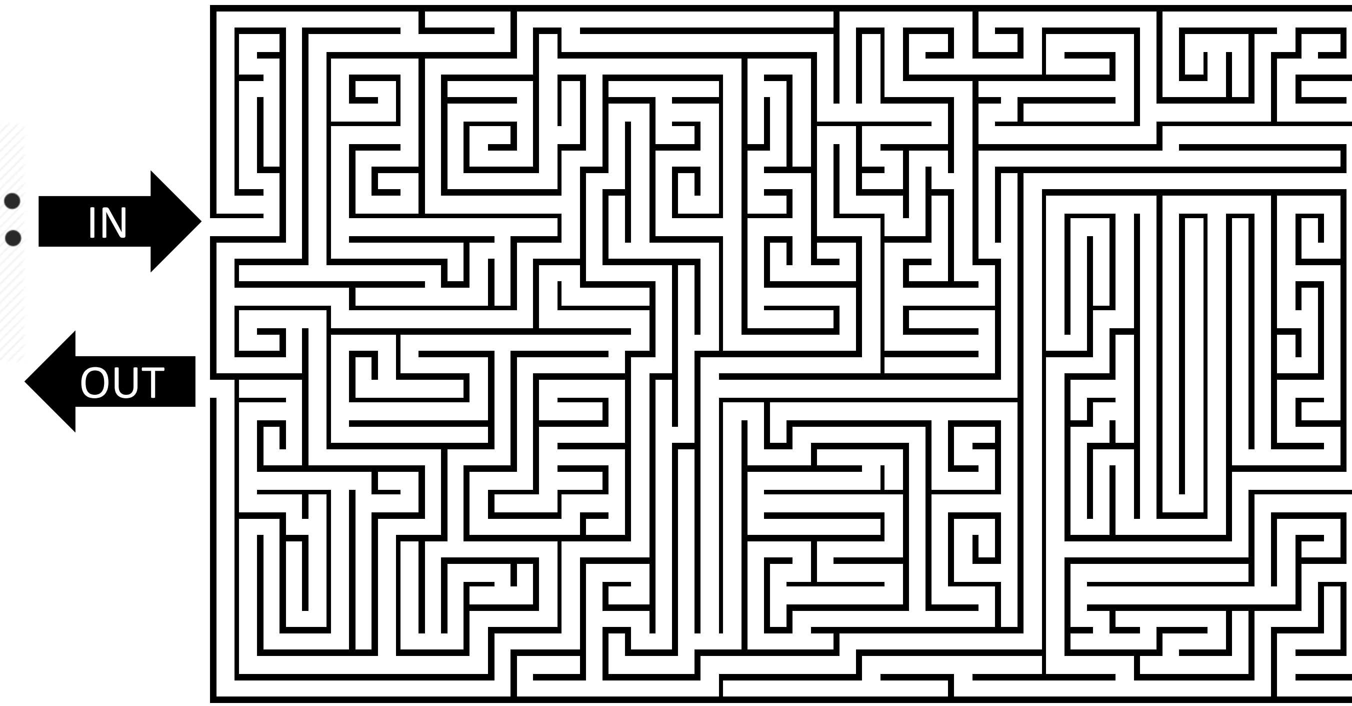
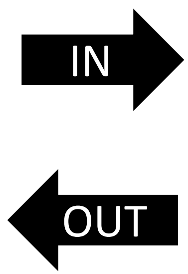
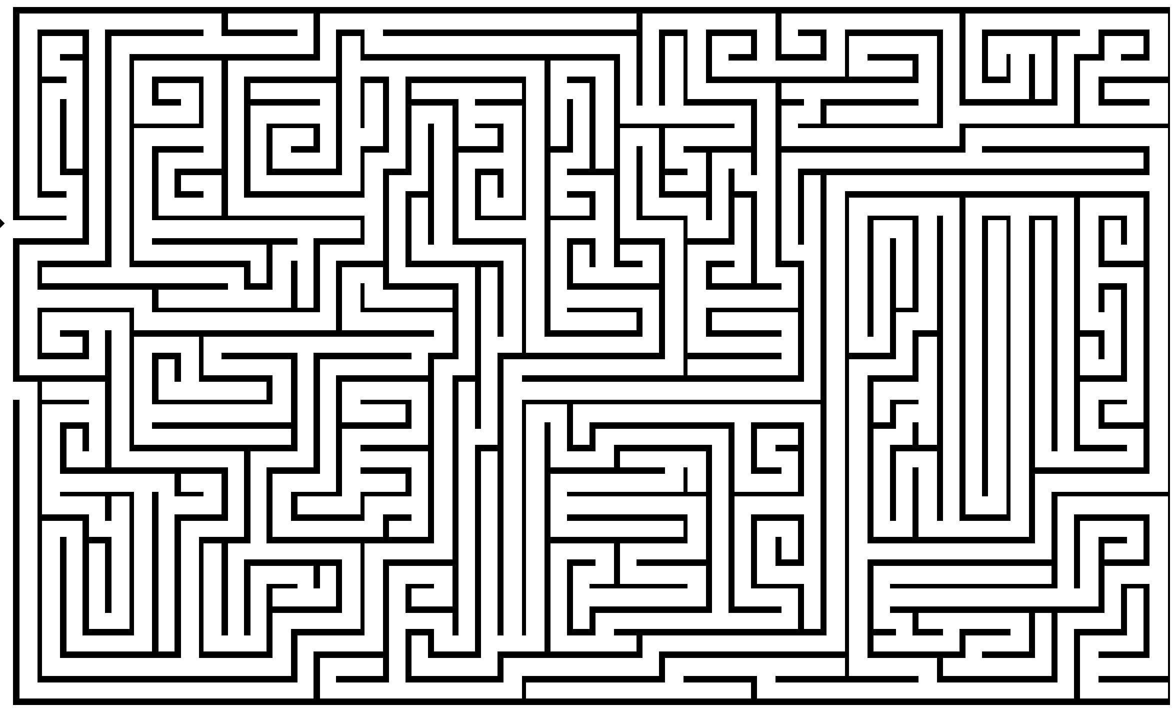


Complete / Systematic

- Searching along **multiple** trajectories (either concurrently or not), eventually covers all the reachable space

Visual example



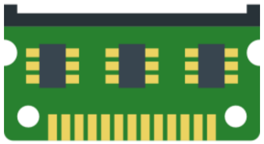


Not complete / Not systematic

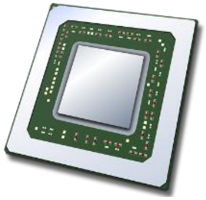
- Searching along a **single** trajectory, eventually gets stuck in a dead end

Space and time complexity

- Space complexity: how does the amount of memory required by the search algorithm grows as a function of the problem's dimension (worst case)?



- Time complexity: how does the time required by the search algorithm grows as a function of the problem's dimension (worst case)?



- Asymptotic trend:
  - We measure complexity with a function $f(n)$ of the input size
  - For analysis purposes, the "Big O" notation is convenient:

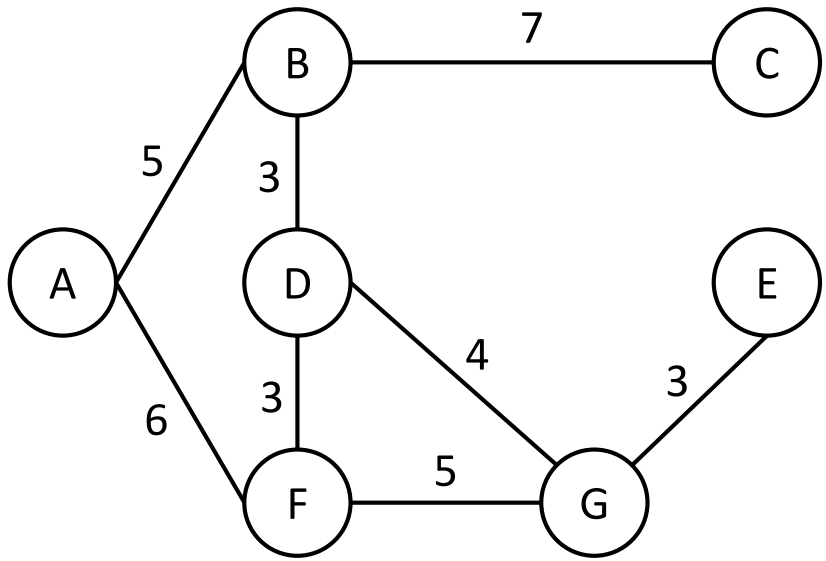
$$\text{A function } f(n) \text{ is } O(g(n)) \text{ if } \exists k > 0, n_0 \text{ such that } f(n) \leq kg(n) \text{ for } n > n_0$$

- An algorithm that is $O(n^2)$ is better than one that is $O(n^5)$
- If $g(n)$ is an exponential, the algorithm is not efficient

Running example

- To present the various search algorithms, we will use this *problem instance* as our running example

State-transition graph:



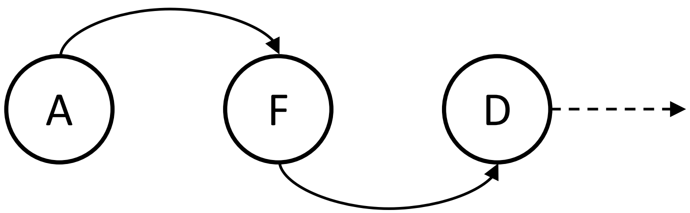
Initial state: A

Desired solution: any path to goal state E

- It might be useful to think it as a map, but keep in mind that this interpretation does not hold for every instance

Search algorithm definition

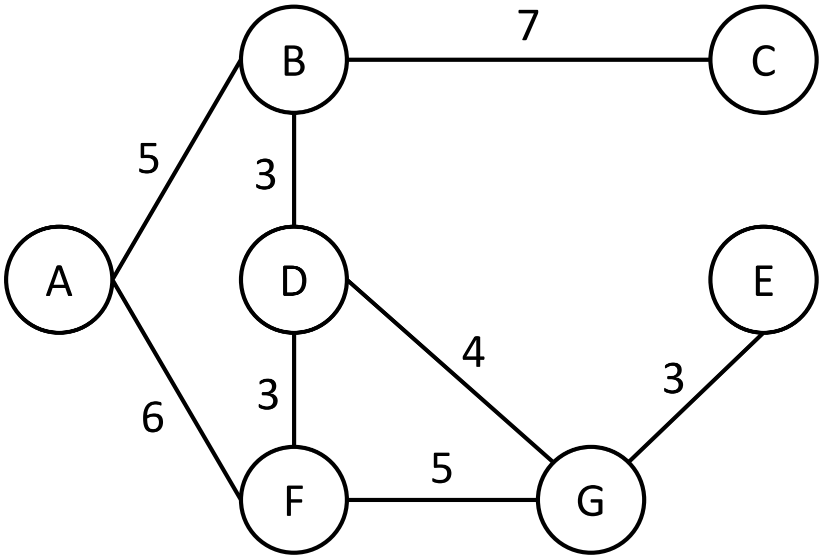
- The different search algorithms are substantially characterized by the answer they provide to the following question:



Given what I searched so far,
where to search next?

- The answer is encoded in a set of rules that drives the search and define its type, let's start with the simplest one

Depth-First Search (DFS)



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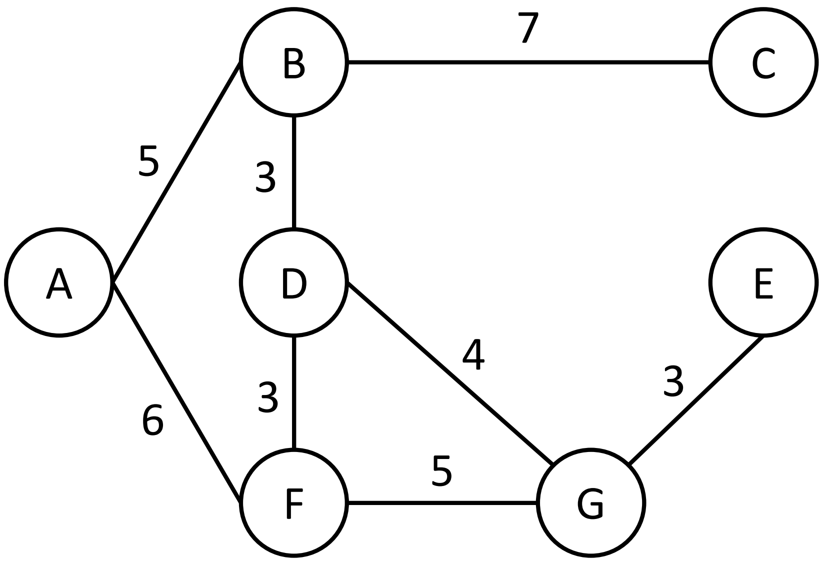

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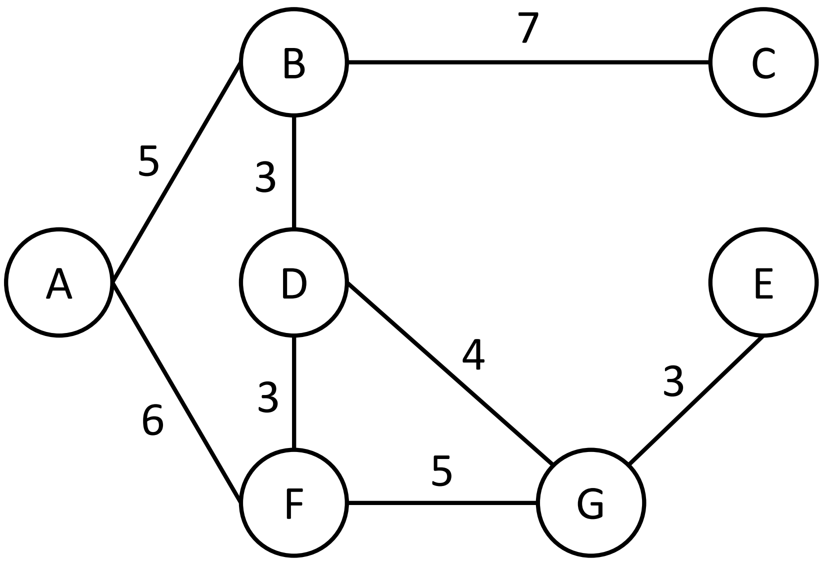

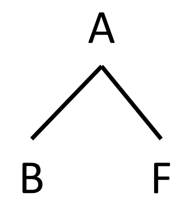

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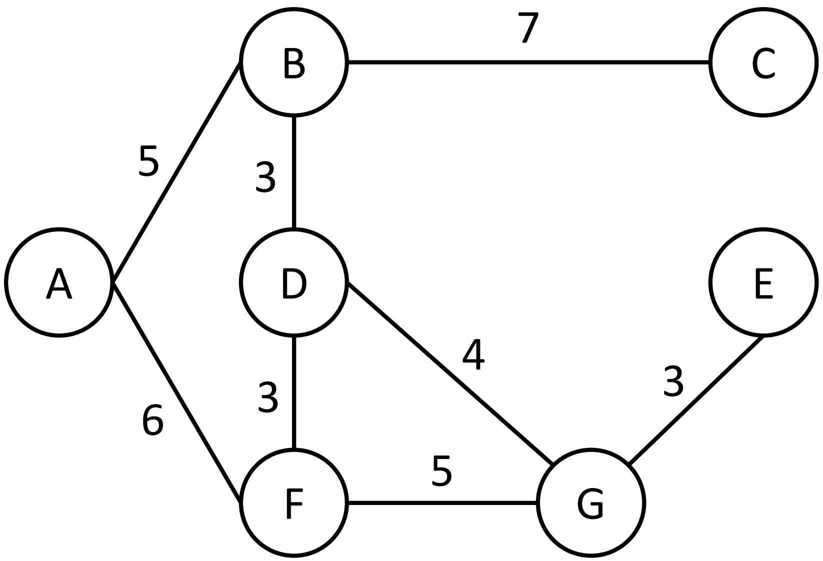

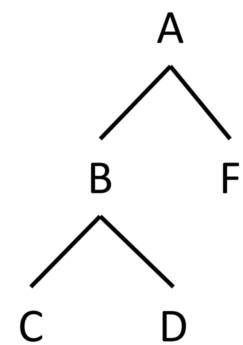

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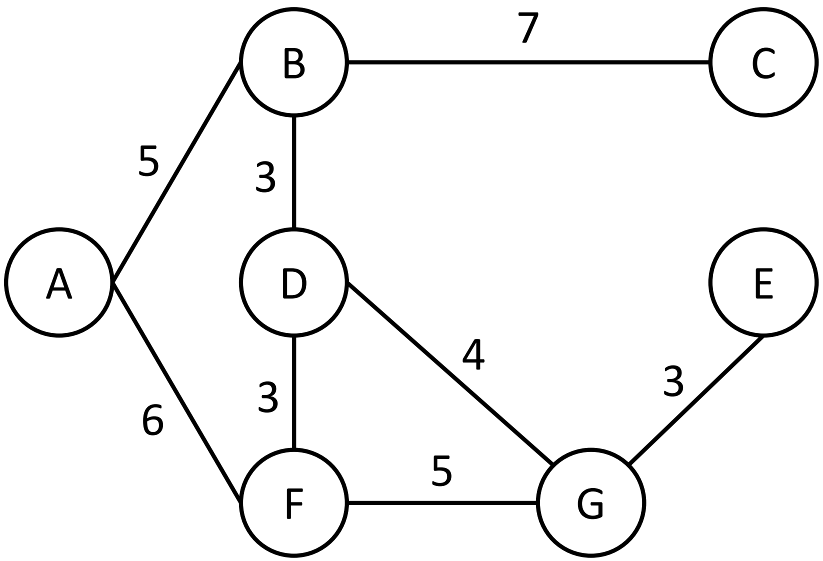

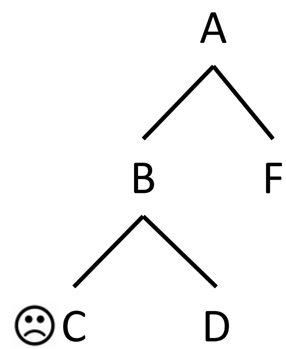

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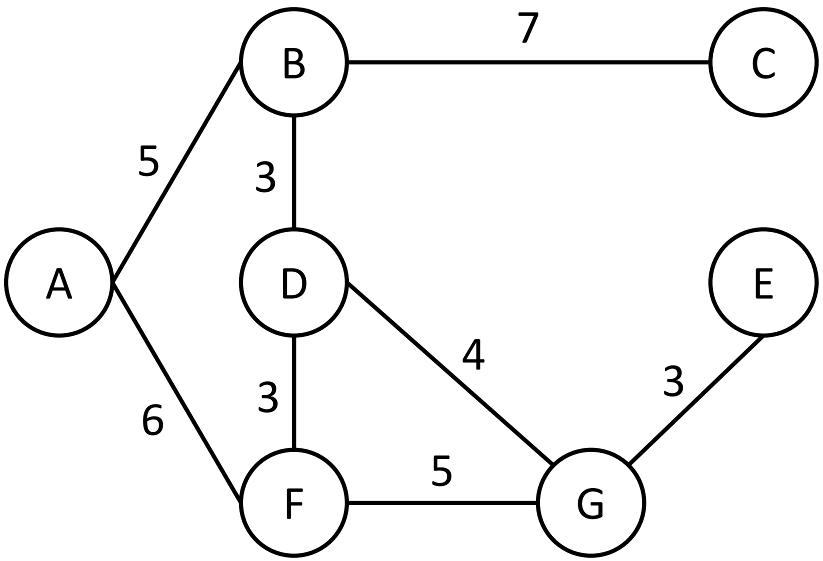

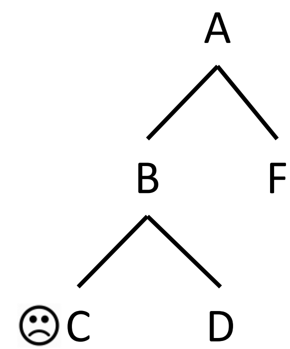

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- We are **avoiding loops** on the same branch (loops are redundant paths)
- A dead end stopped the search, DFS seems not complete. Can we fix this?
- Let's endow our DFS with **backtracking**: a way to reconsider previously evaluated decisions

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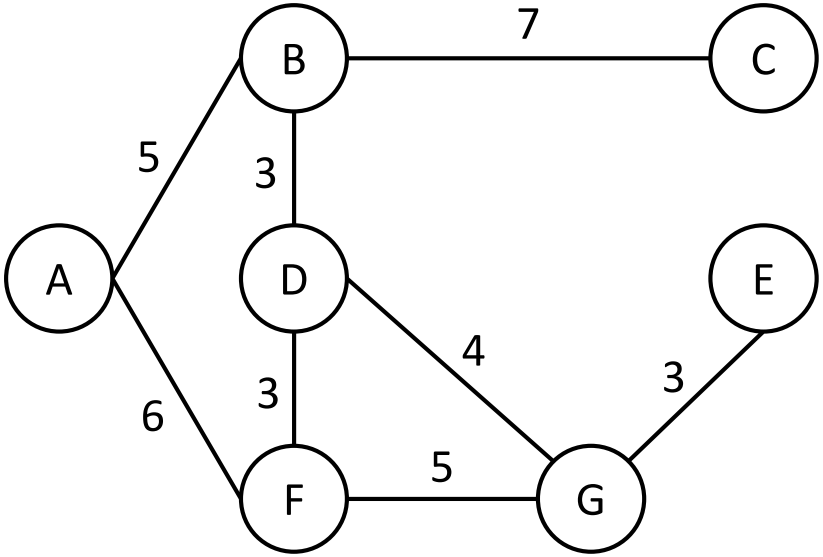

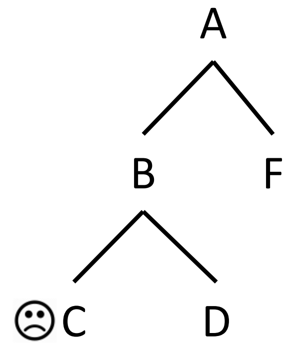

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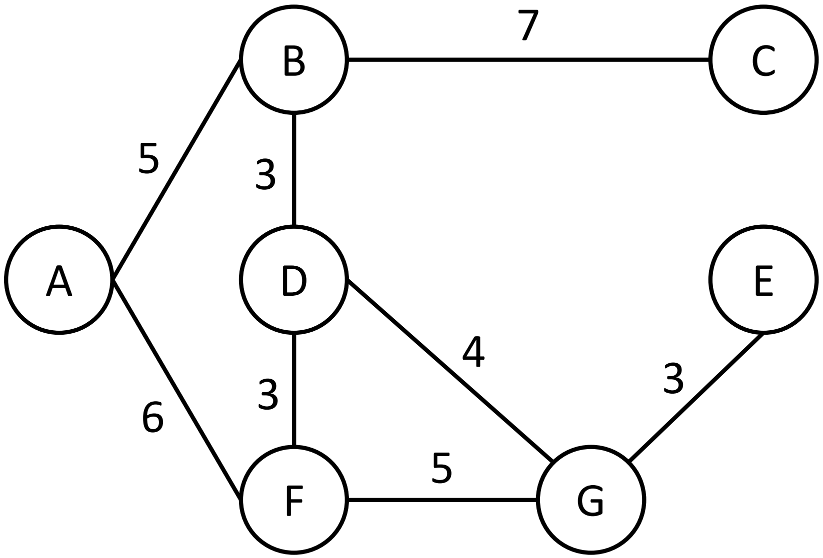

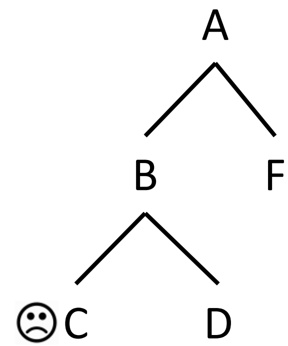

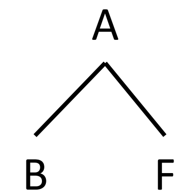

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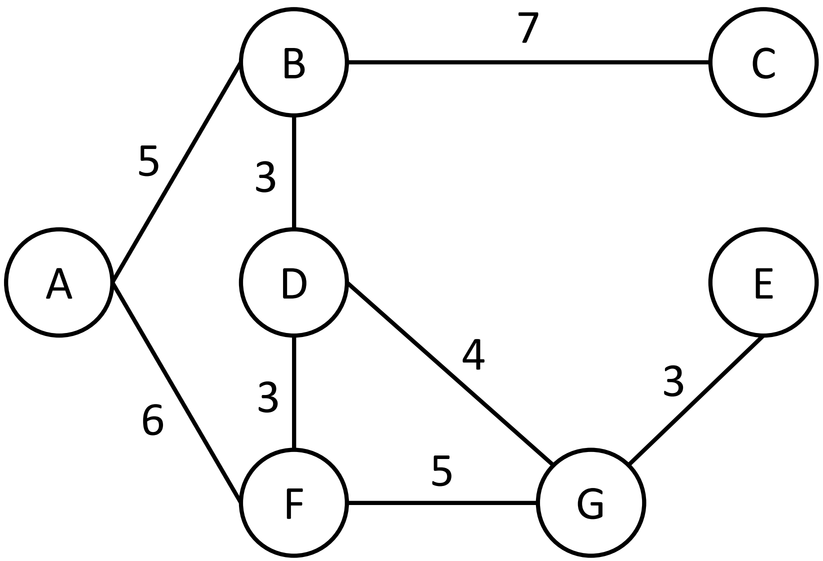

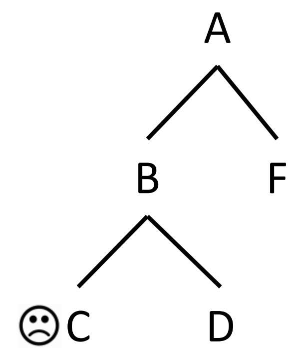

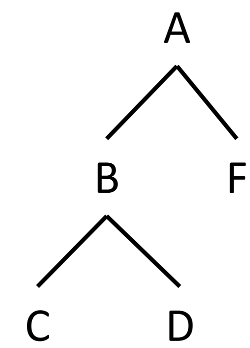

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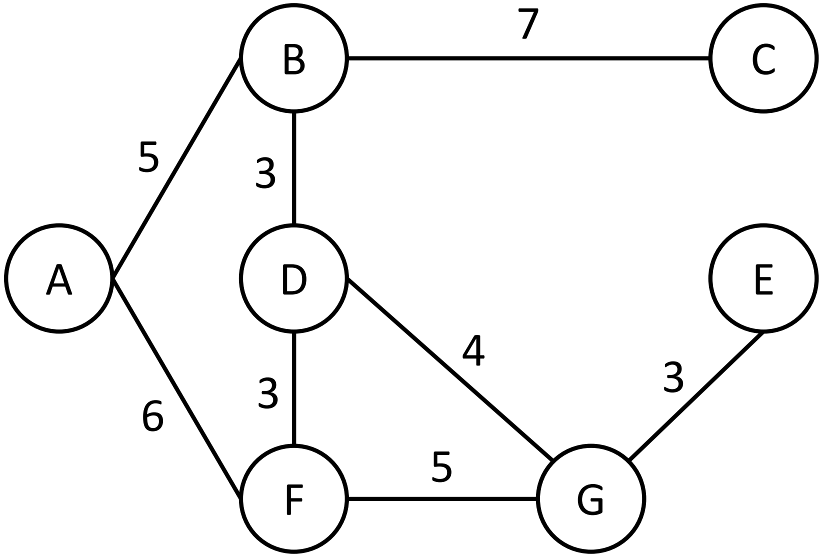

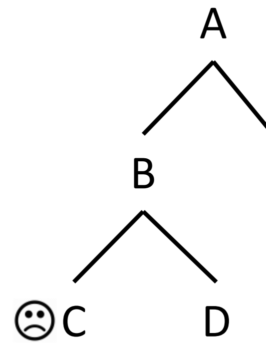

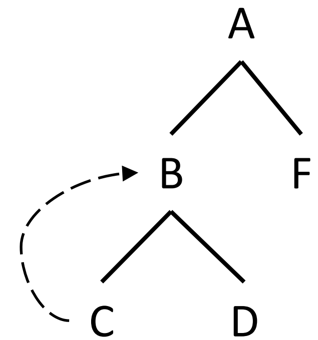

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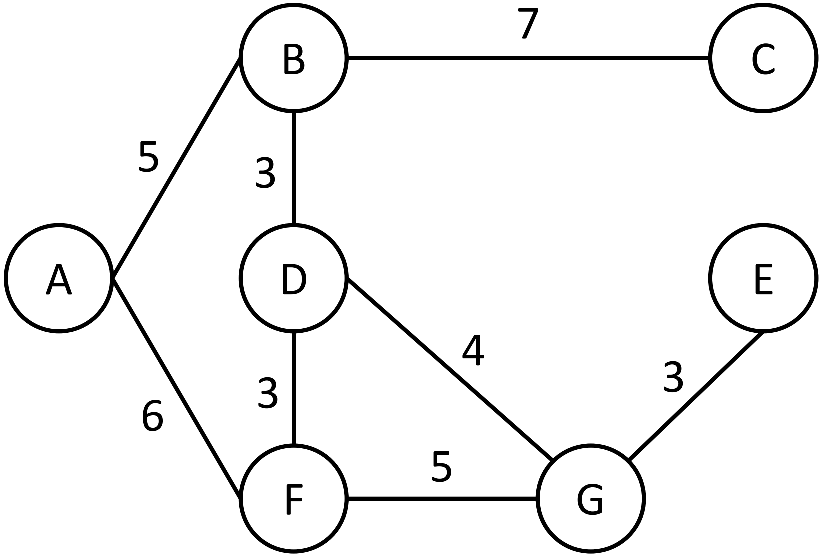

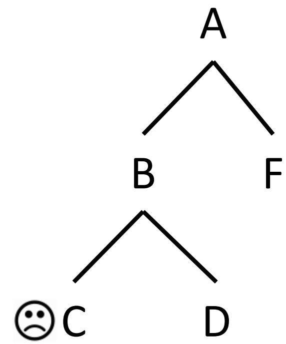

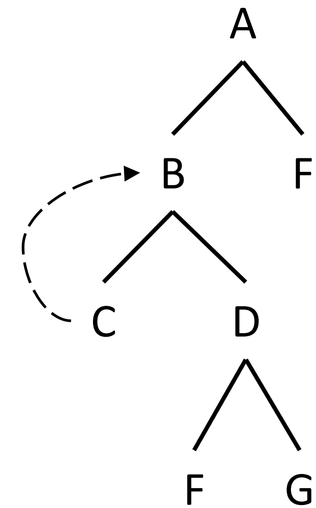

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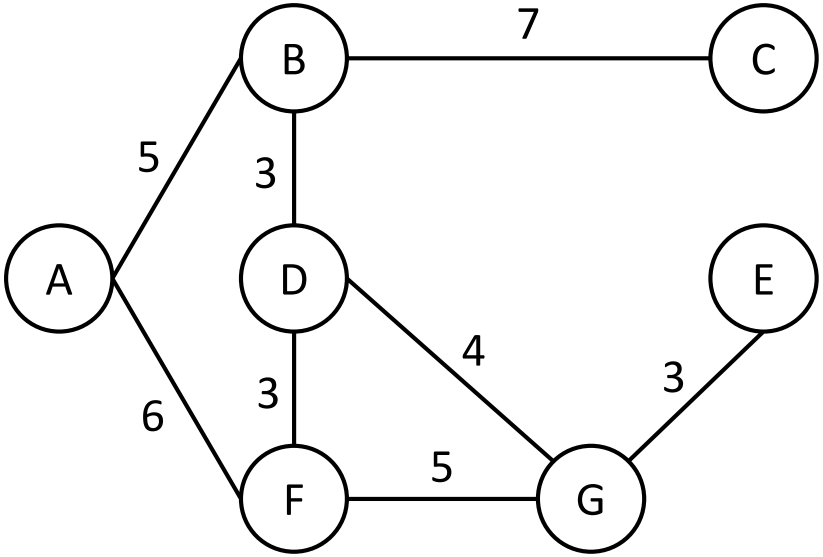

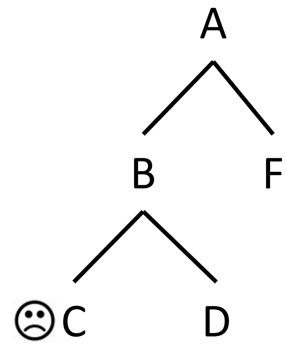

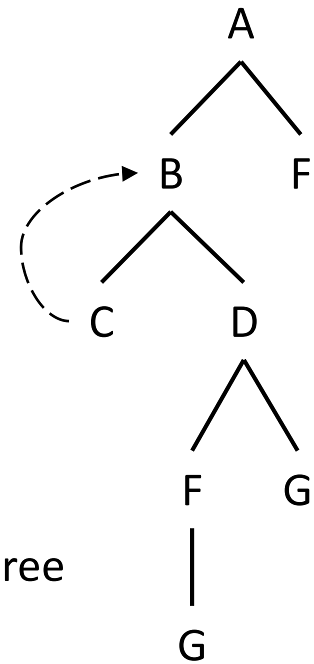

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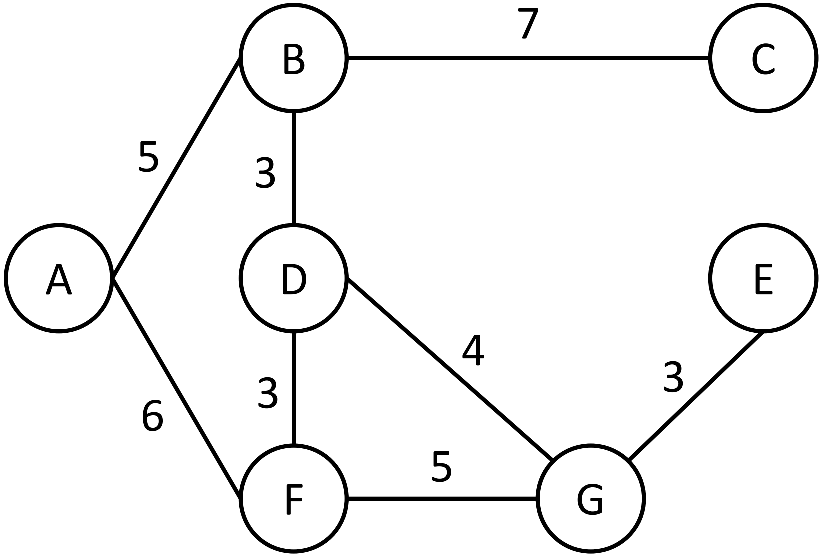

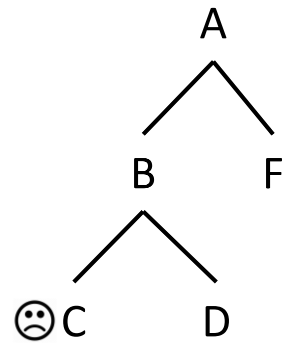

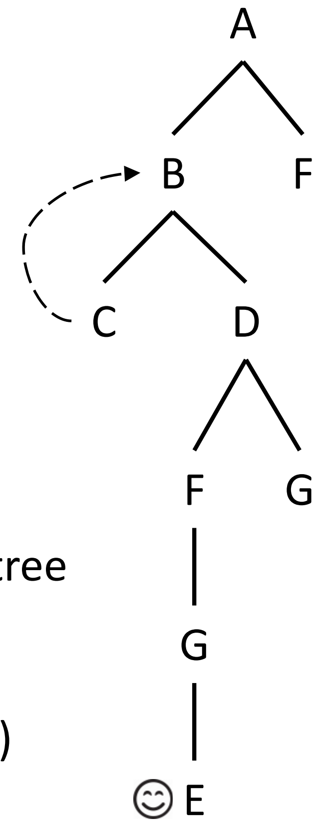

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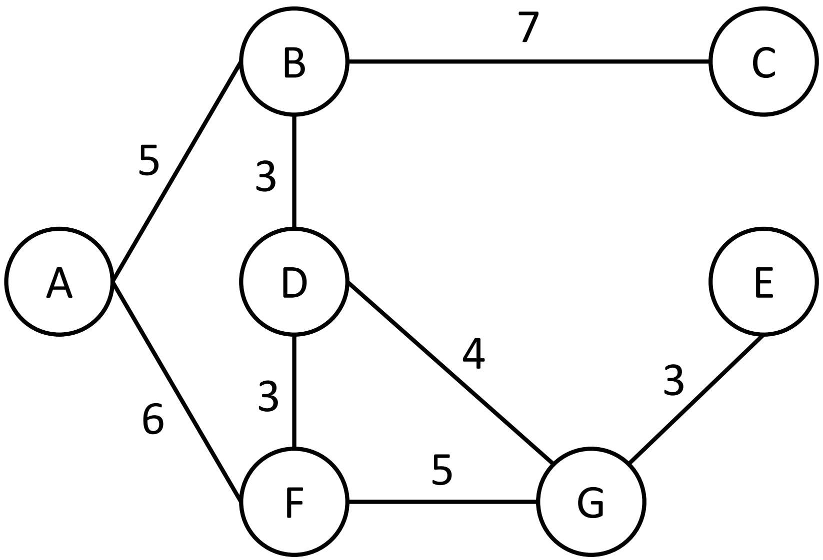
Solution: (A->B->D->F->G->E)

Depth-First Search (DFS)

- DFS with loops removal and BT is sound and complete
- Call b the maximum branching factor, i.e., the maximum number of actions available in a state
- Call d the maximum depth of a solution, i.e., the maximum number of actions in a path
- Space complexity: $O(d)$
- Time complexity: $1 + b + b^2 + \ldots + b^d = O(b^d)$

Breadth-First Search (BFS)



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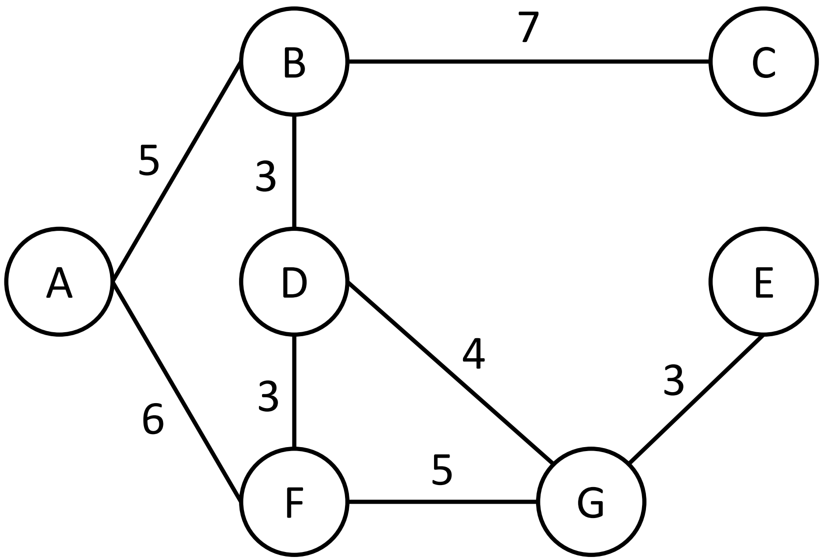

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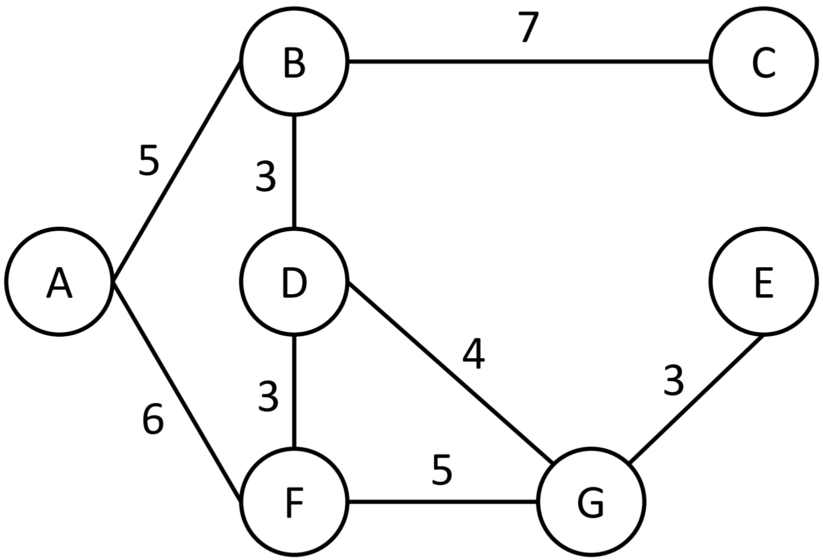

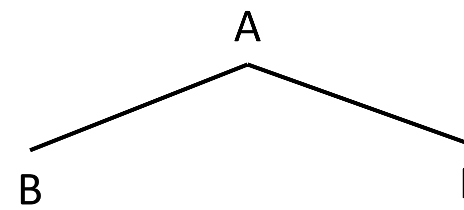

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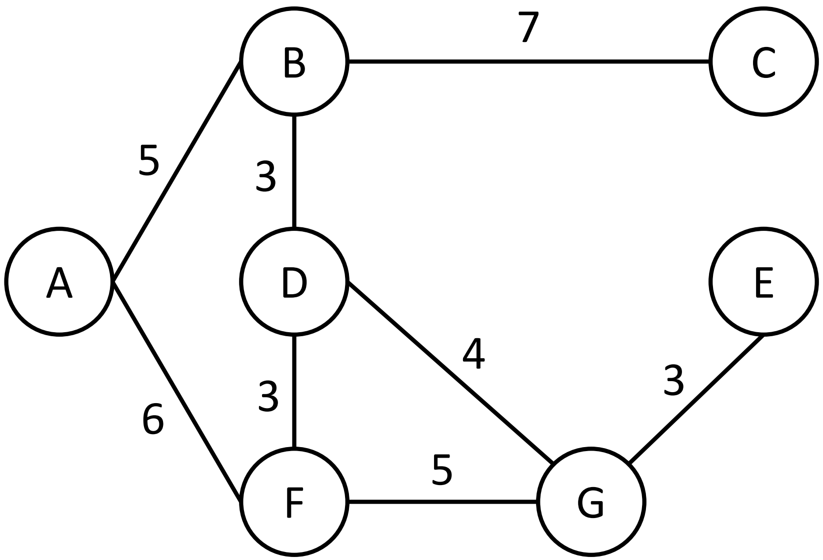
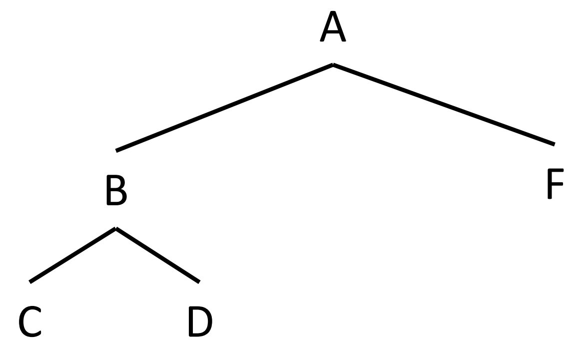
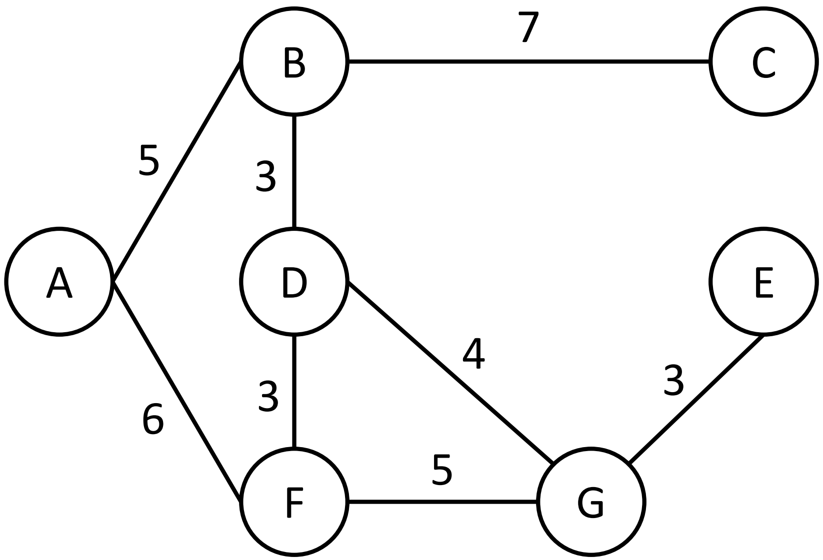
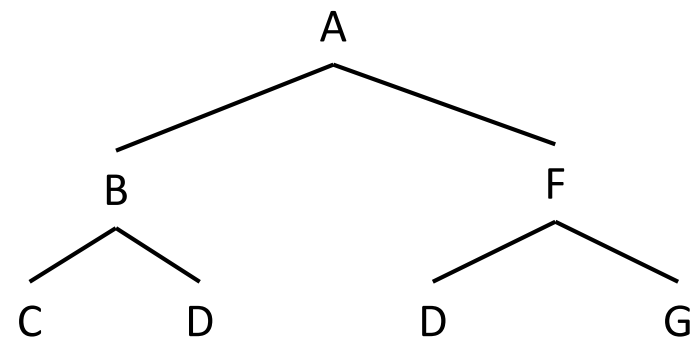




Breadth-First Search (BFS)





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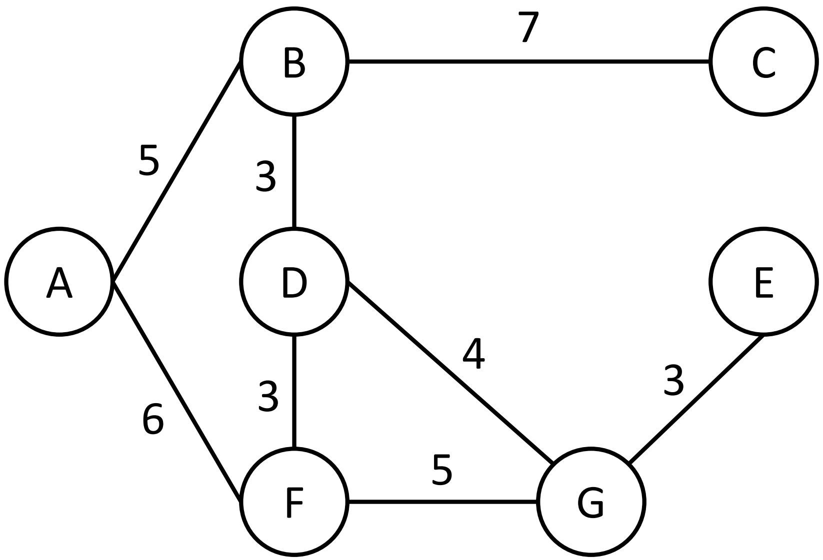

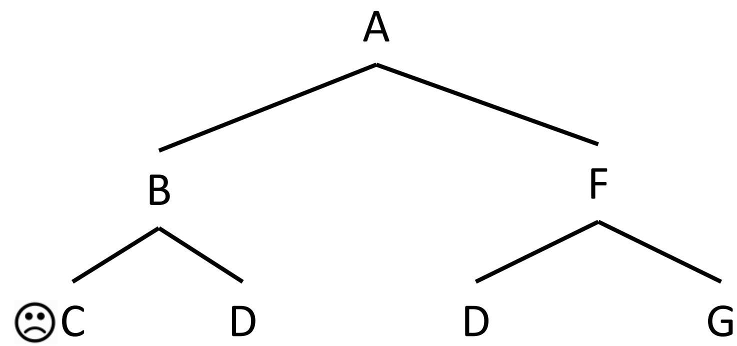

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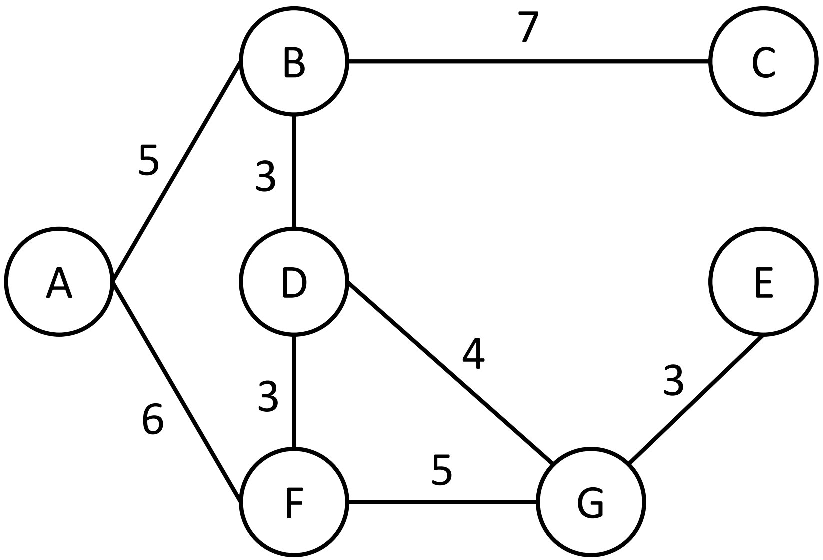

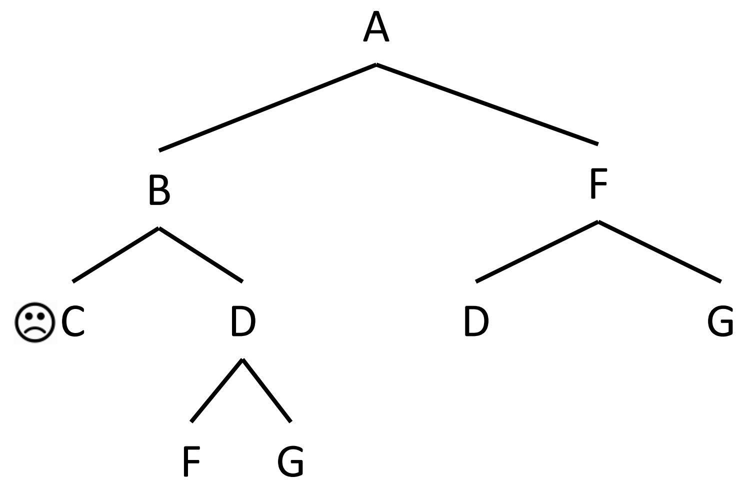

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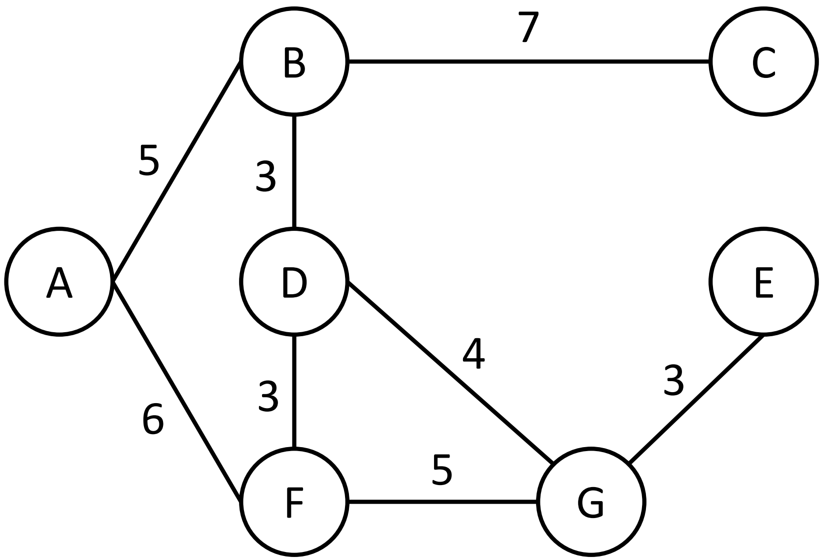

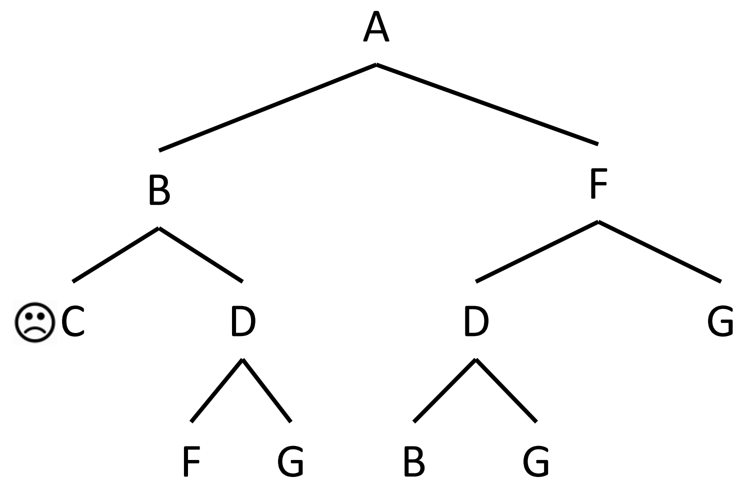

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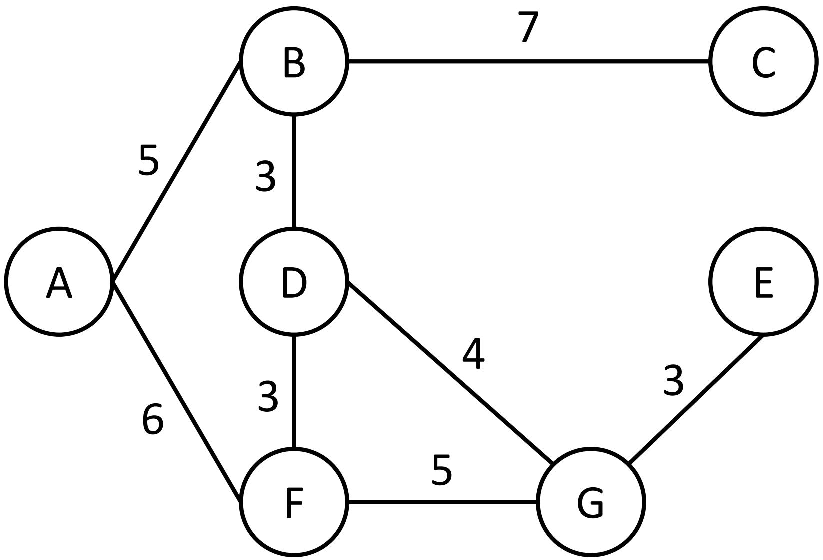
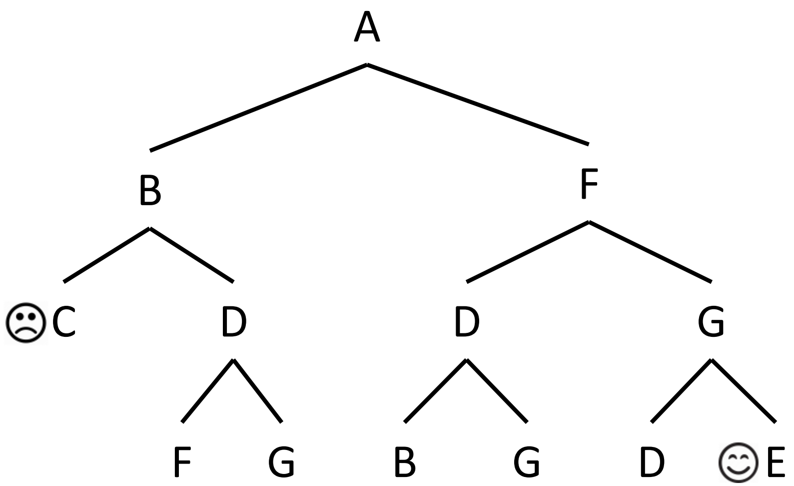
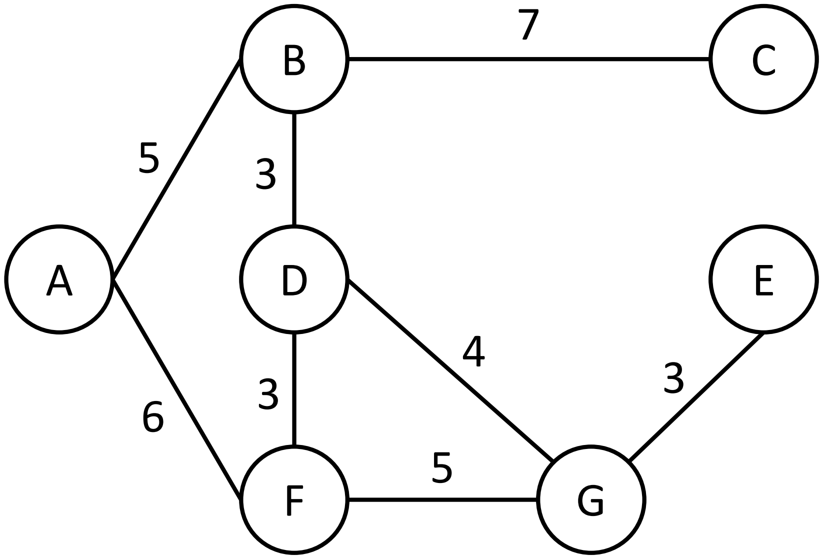
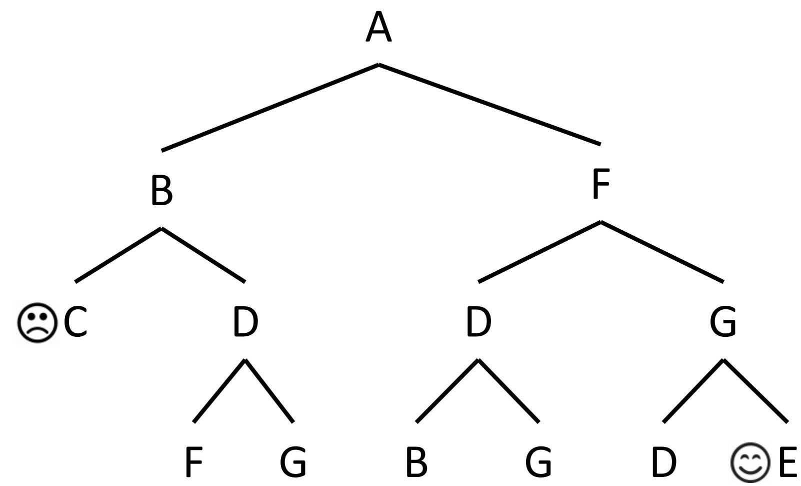




Solution: (A->F->G->E)

Breadth-First Search (BFS)



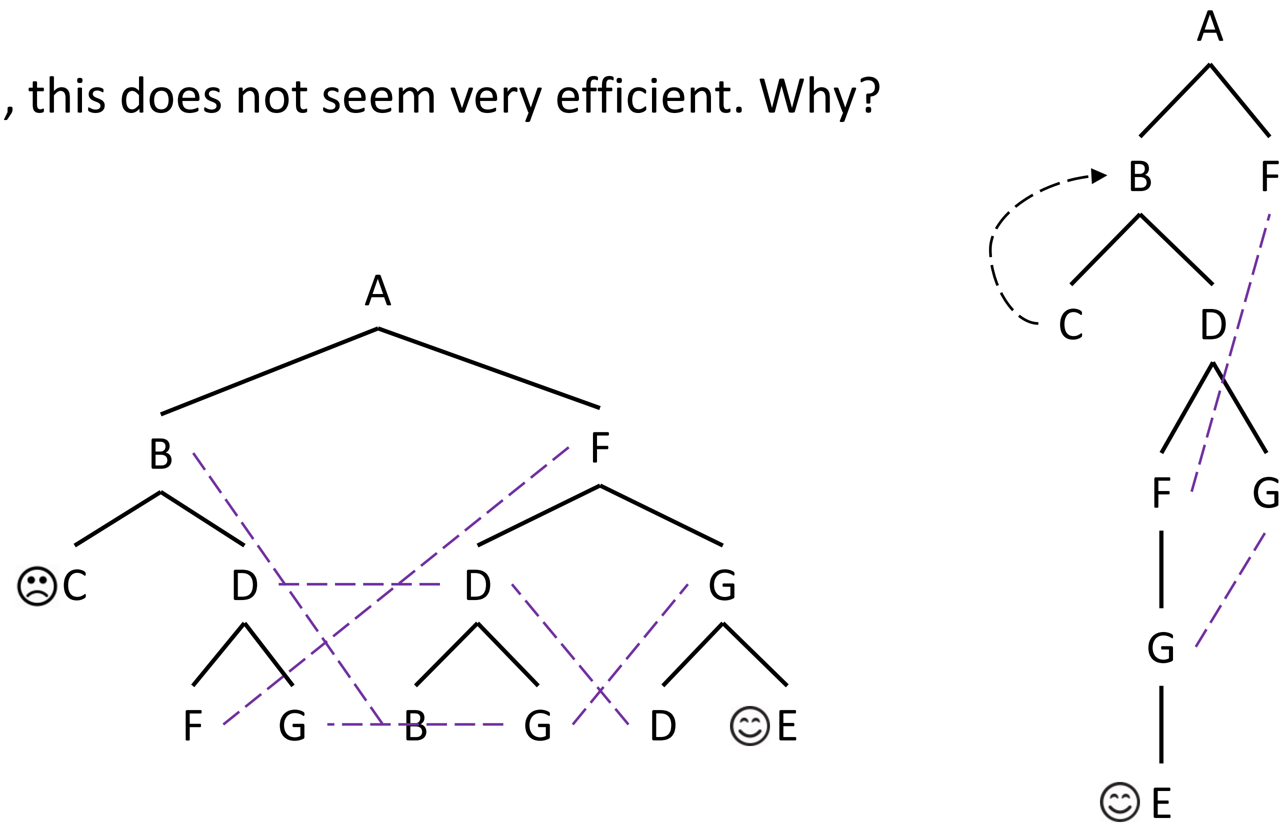


Solution: (A->F->G->E)

- A Breadth-First Search (BFS) chooses the shallowest node, thus exploring in a level by level fashion
- It has a more conservative behavior and does not need to reconsider decisions
- Call q the depth of the shallowest solution (in general $q \leq d$)
- Space complexity: $O(b^q)$
- Time complexity: $O(b^q)$

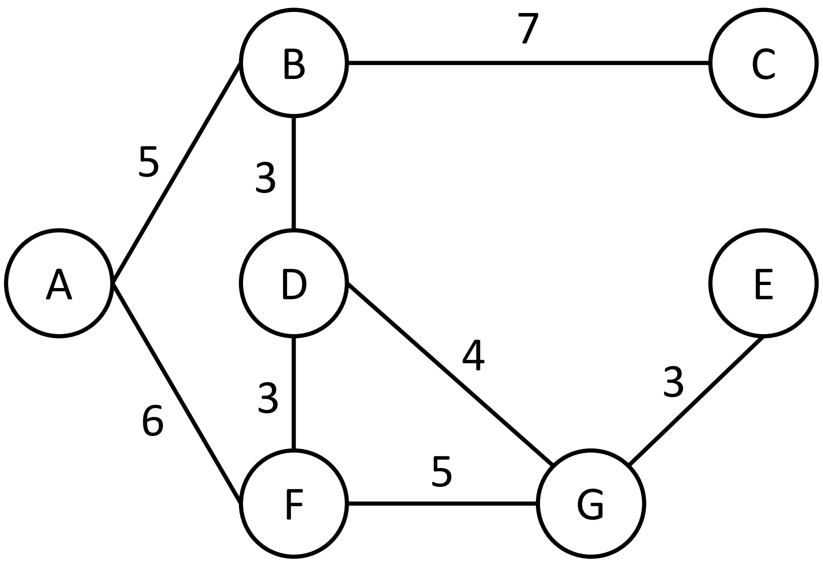
Redundant paths

- Both DFS and BFS visited some nodes multiple times (avoiding loops prevents this to happen only within the same branch)
- In general, this does not seem very efficient. Why?

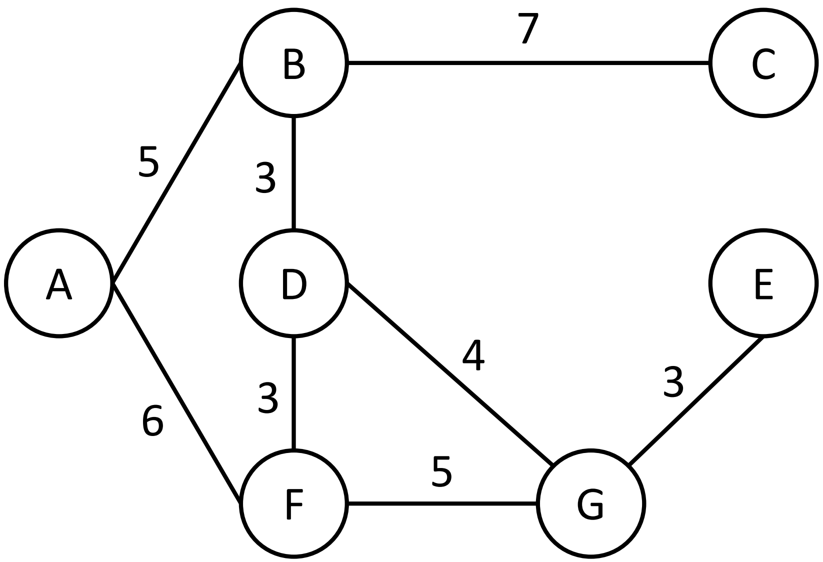


- Idea: discard a newly generated node if already present somewhere on the tree, we can do this with an **enqueued list**

DFS with Enqueued List



DFS with Enqueued List



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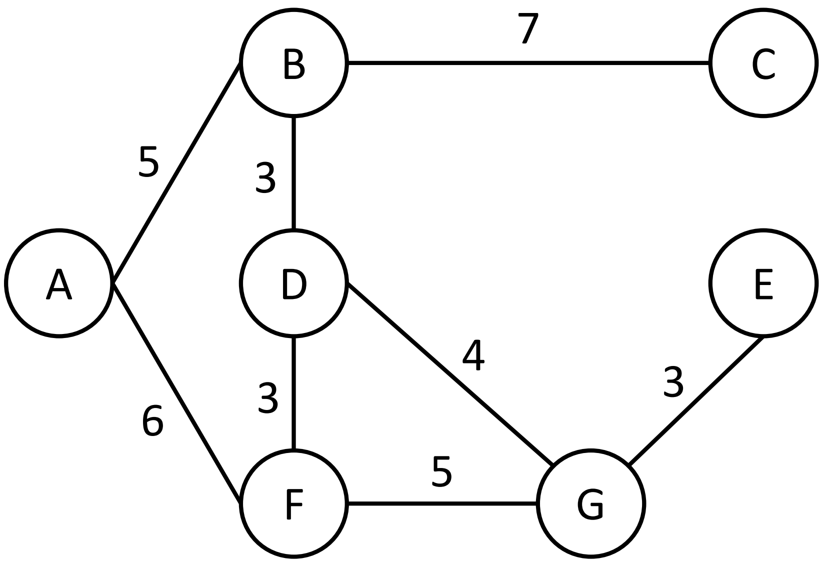
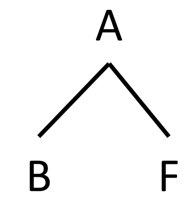
DFS with Enqueued List





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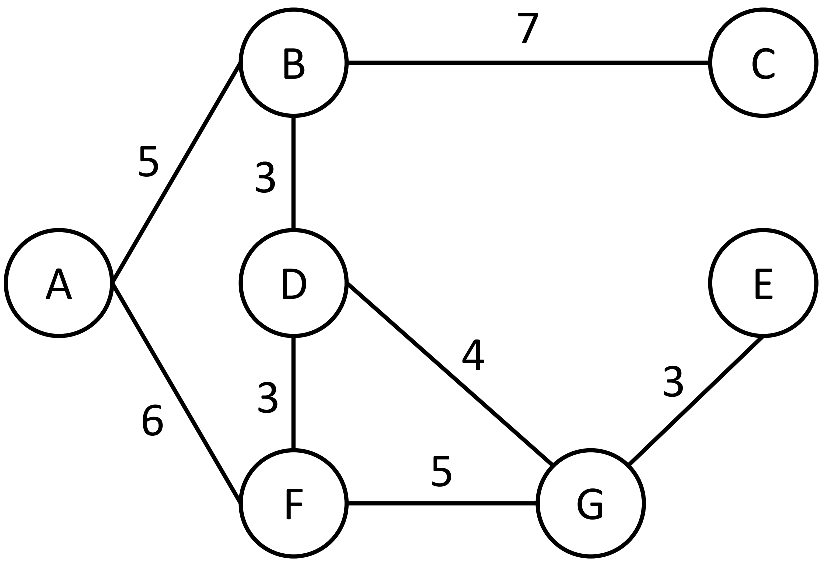

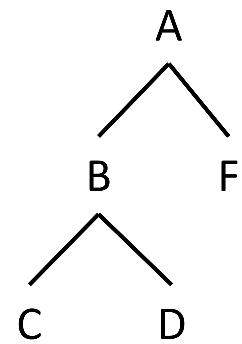

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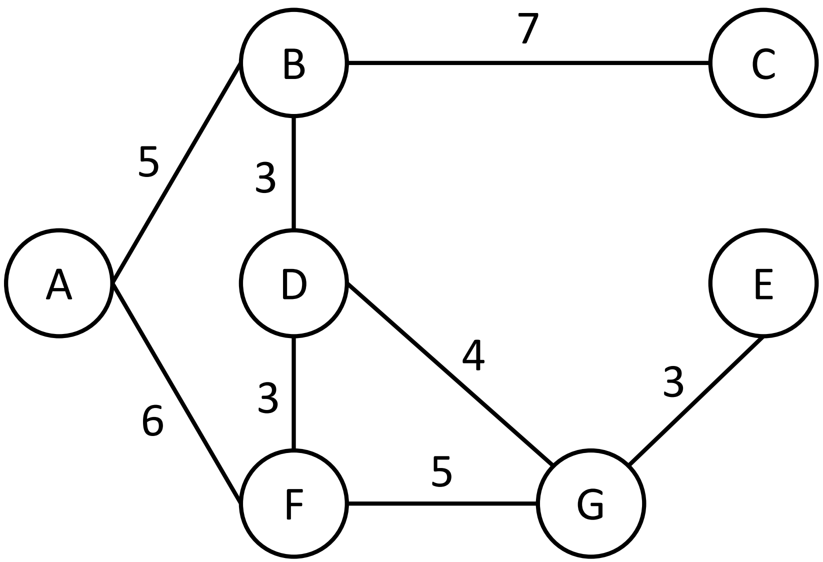
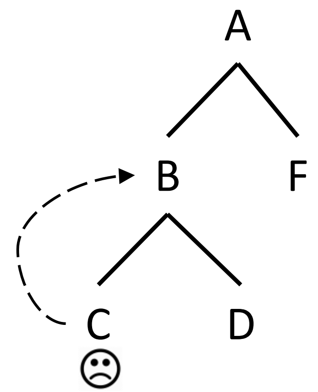
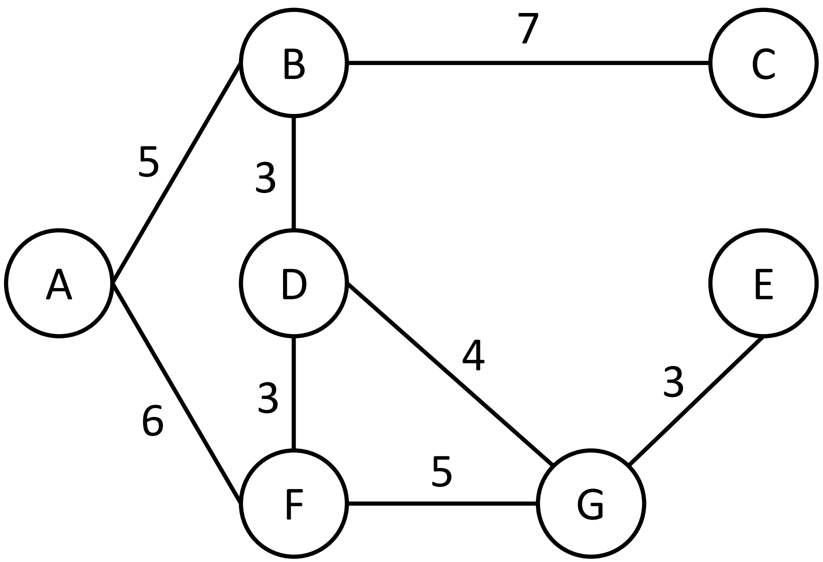
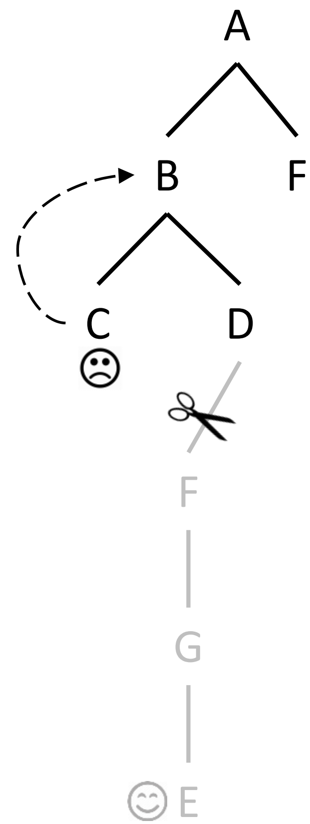




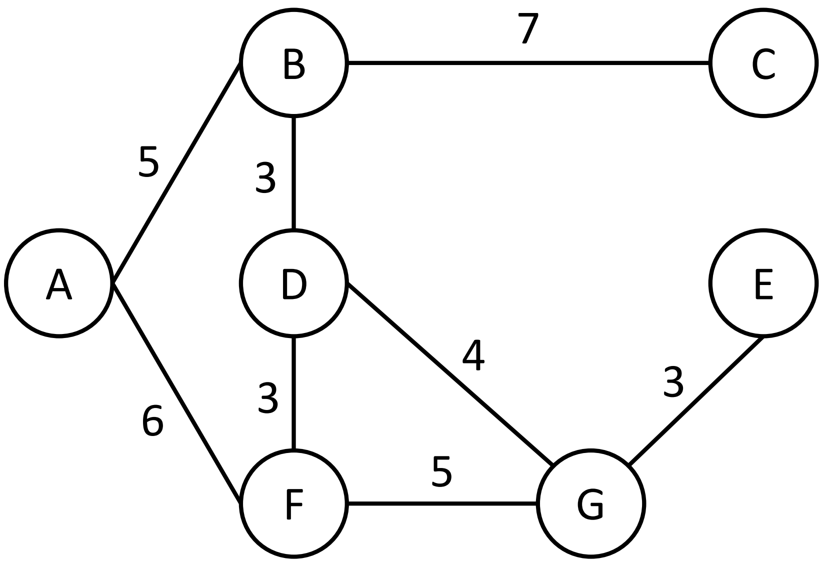
DFS with Enqueued List

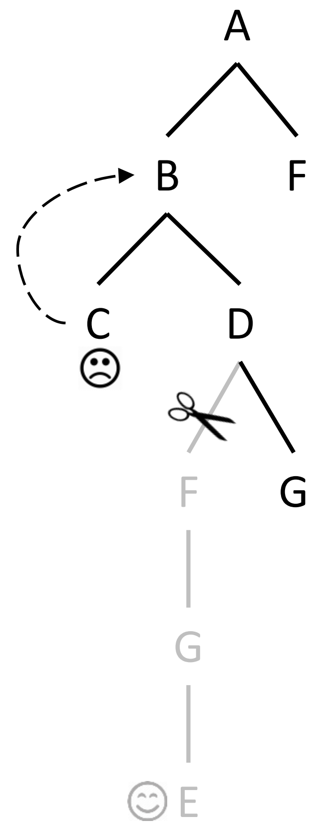




- Node F ha already been "enqueued" on the tree, by discarding it we *prune* a branch of the tree

DFS with Enqueued List





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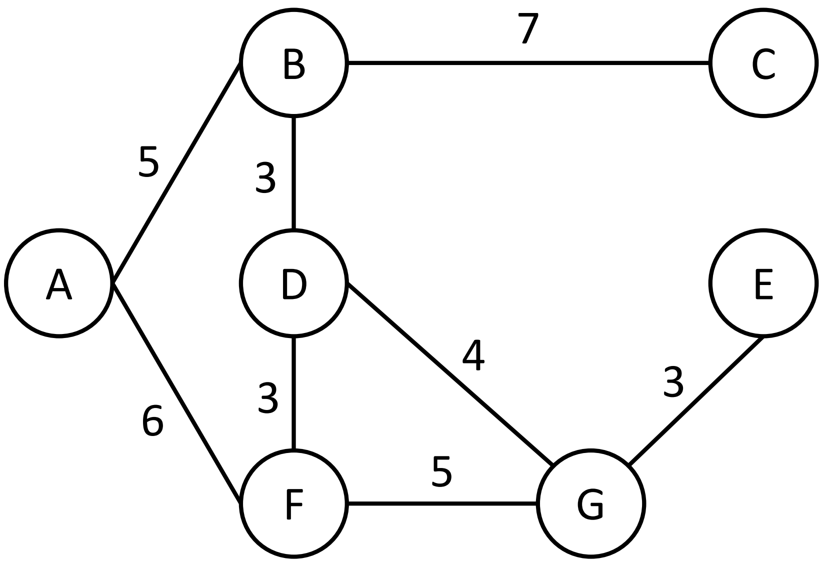

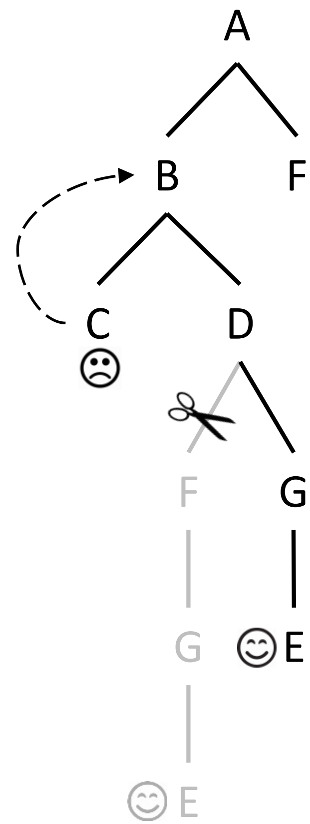

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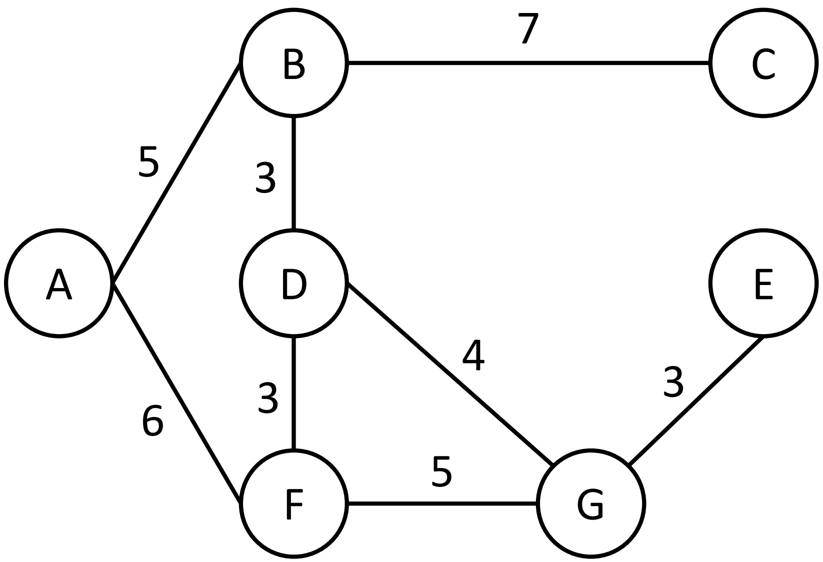
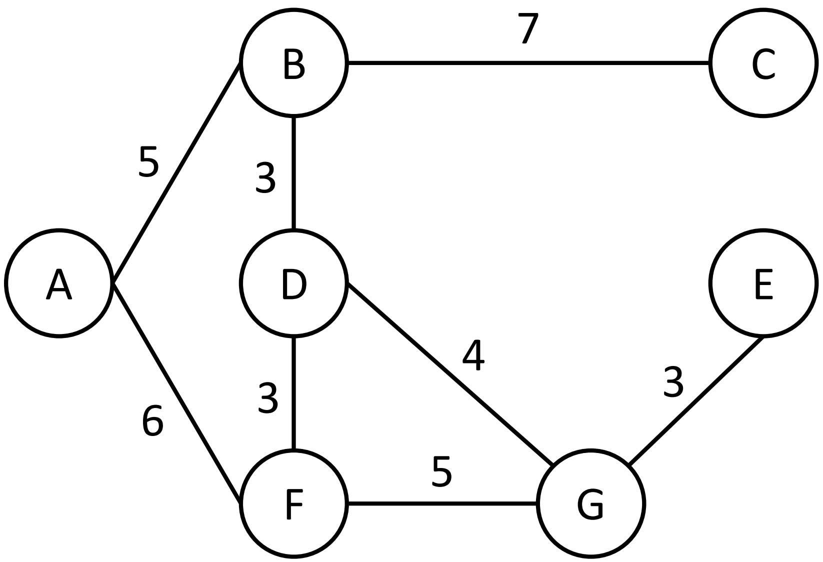
BFS with Enqueued List



BFS with Enqueued List



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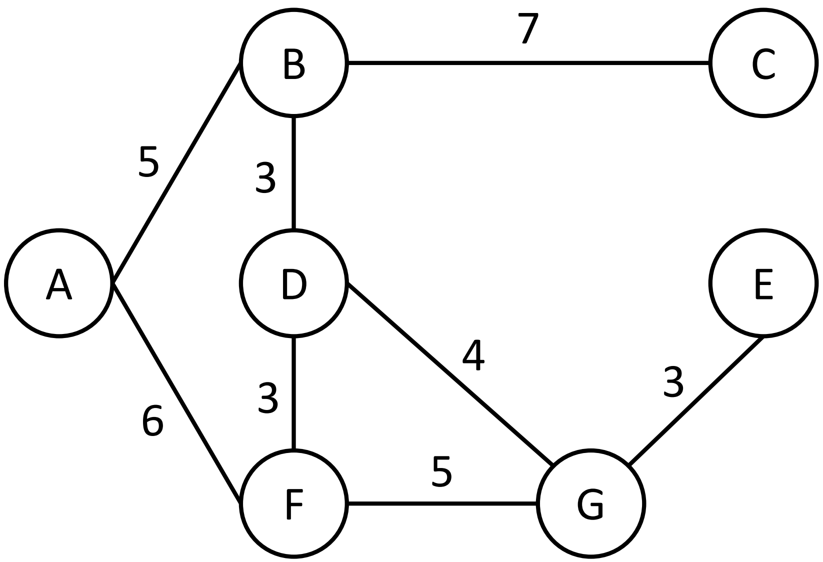
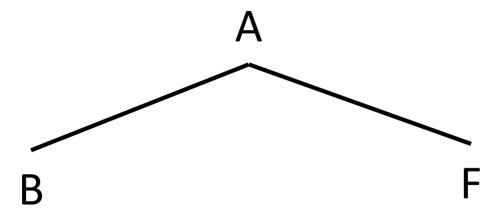
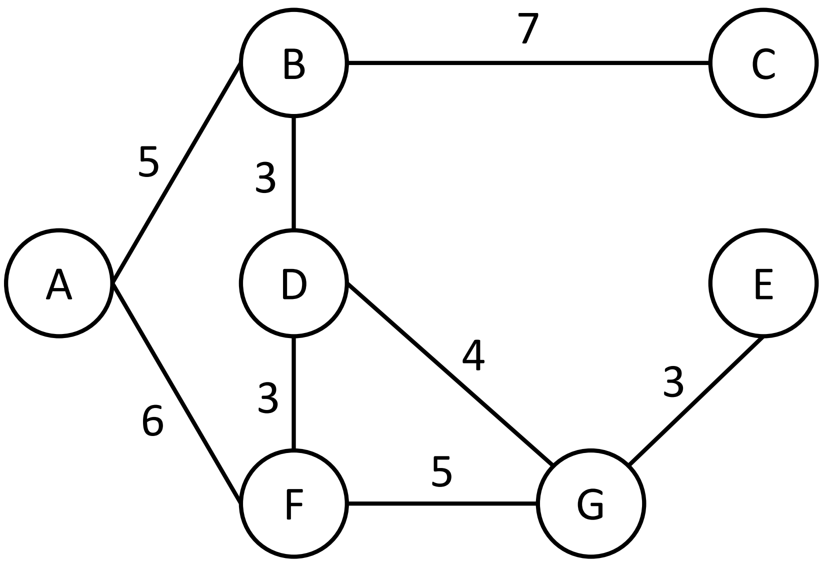
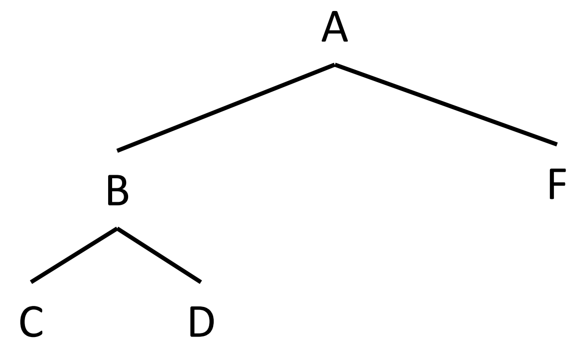
BFS with Enqueued List



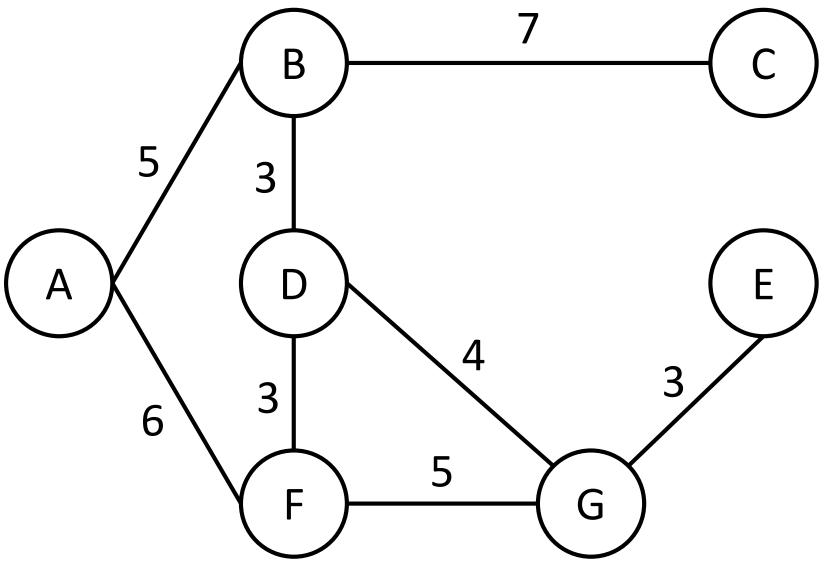


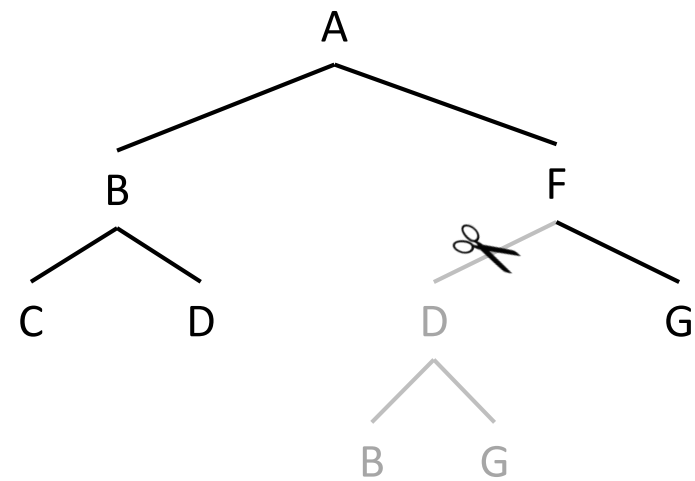
BFS with Enqueued List



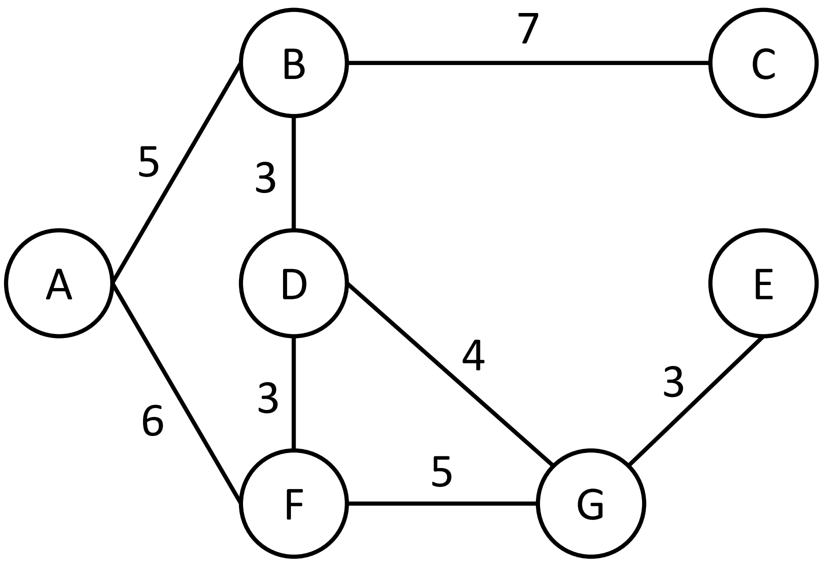


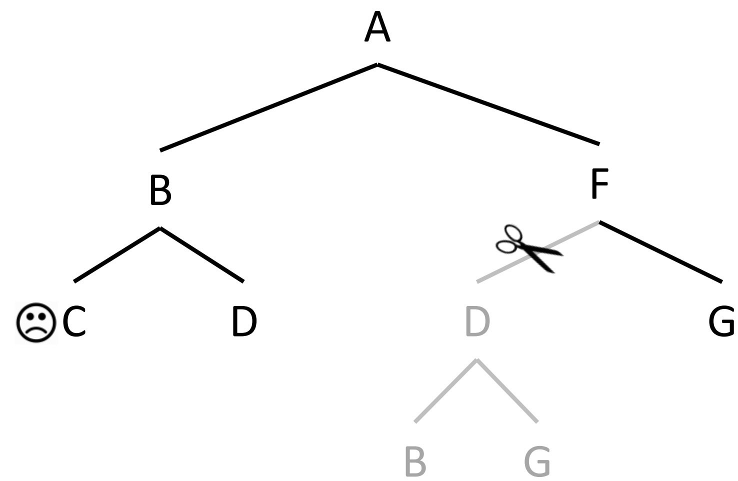
BFS with Enqueued List



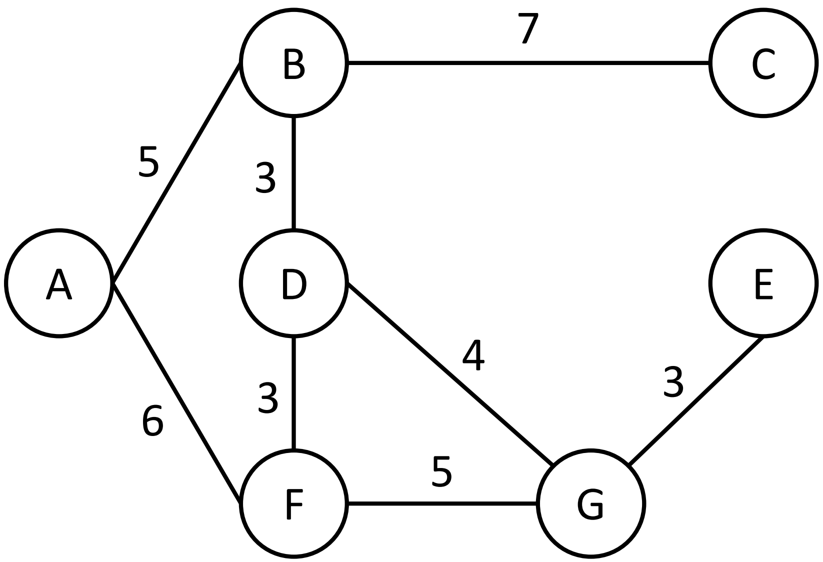


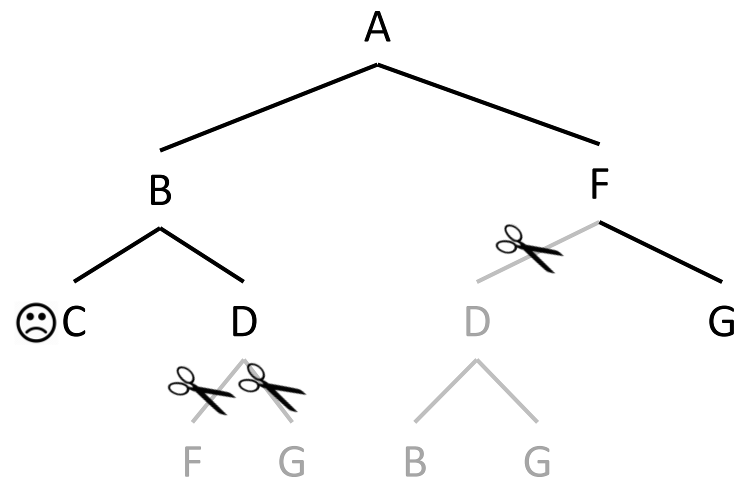
BFS with Enqueued List



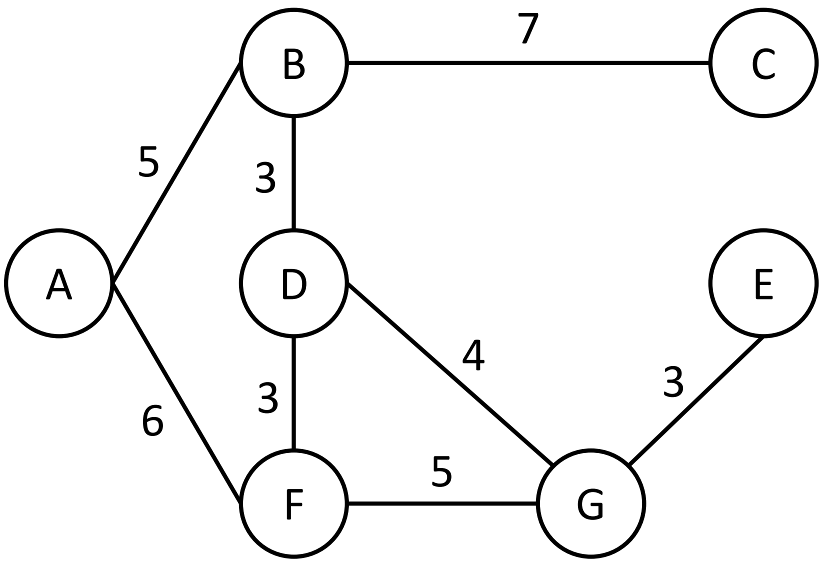


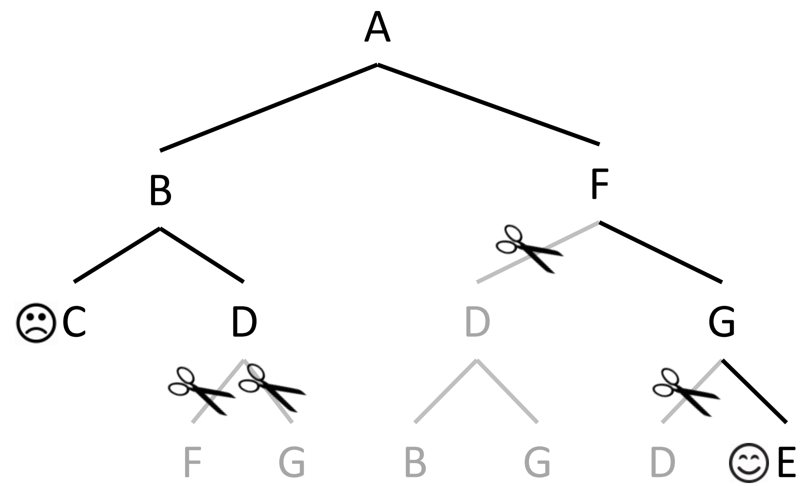
BFS with Enqueued List





BFS with Enqueued List

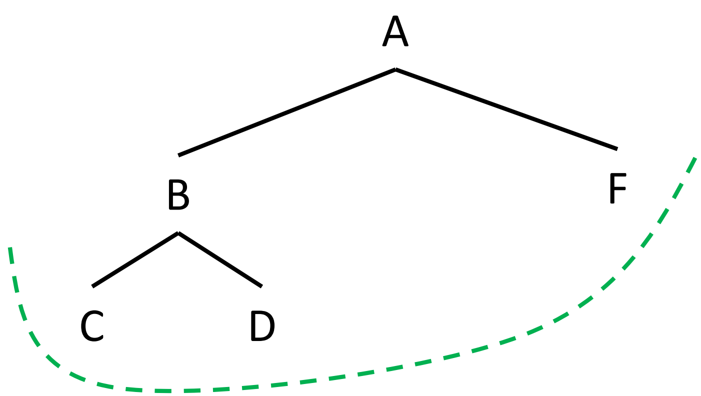


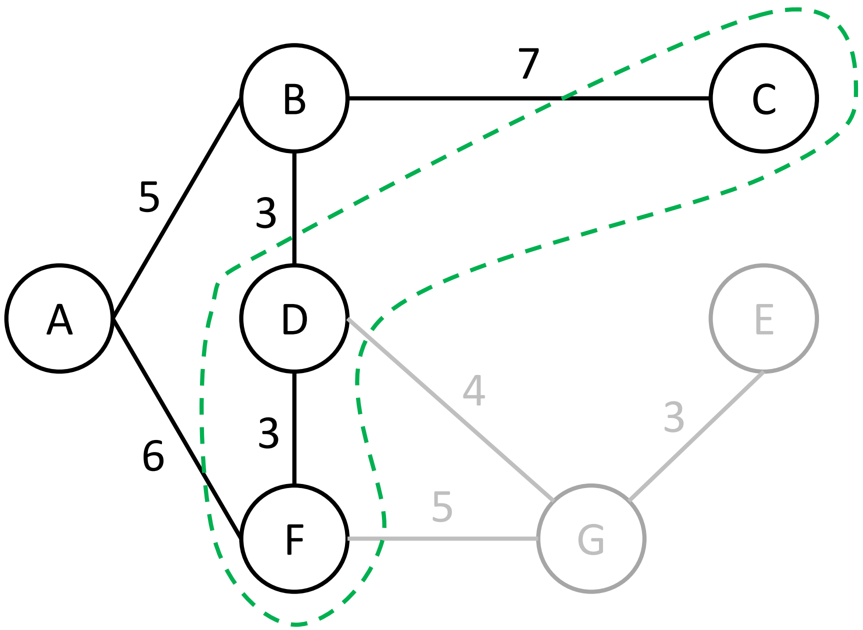


Implementation

- The implementation of the previous algorithms is based on two data structures:
  - A queue **F** (Frontier), elements ordered by priority, a selection consumes the element with highest priority
  - A list **EL** (Enqueued List, nodes that have already been put on the tree)

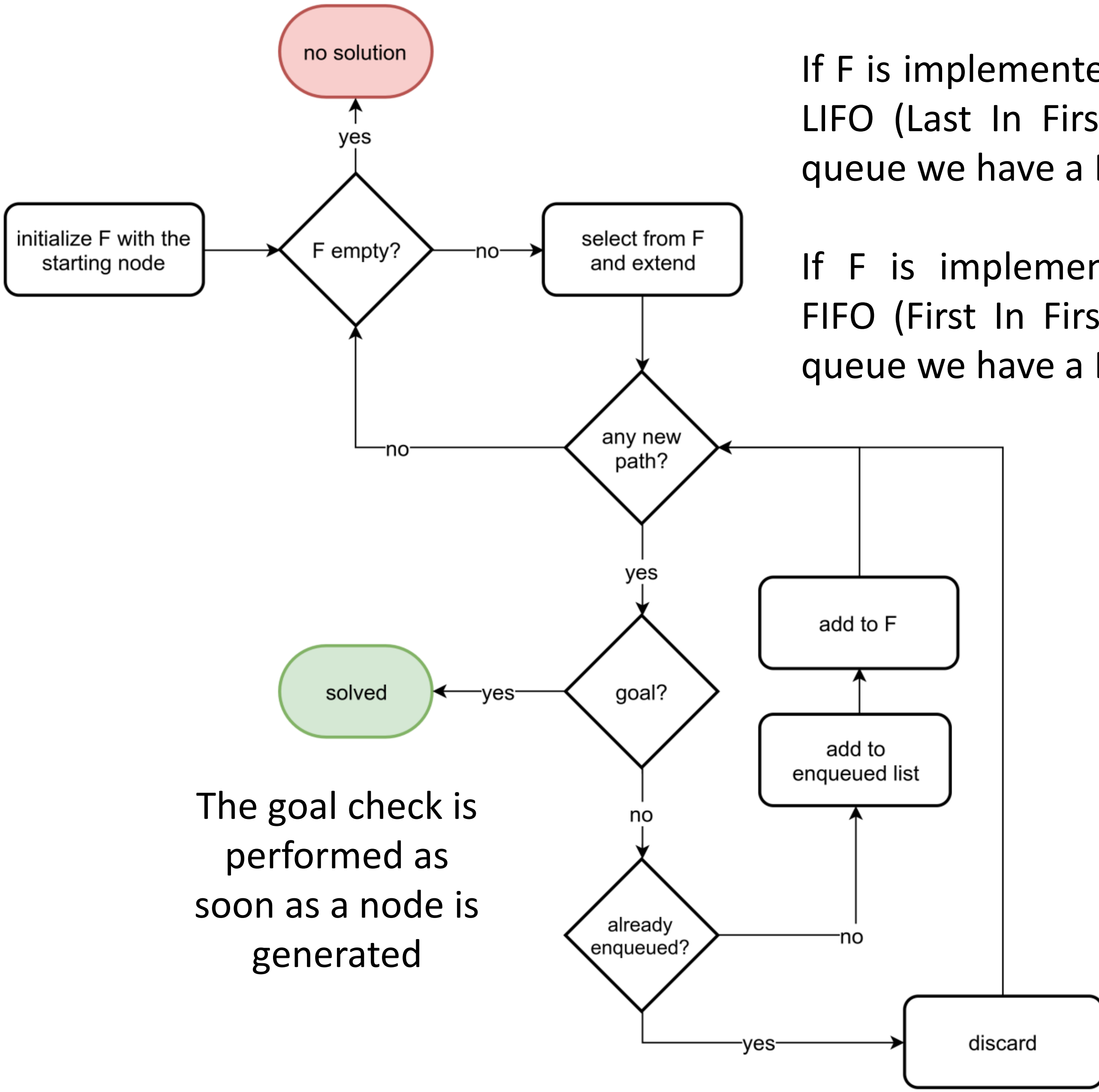
- The frontier F contains the terminal nodes of all the paths currently under exploration on the tree





- The frontier **separates** the explored part of the state space from the unexplored part
- In order to reach a state that we still did not searched, we need to pass from the frontier (separation property)

Implementation



If F is implemented as a LIFO (Last In First Out) queue we have a DFS

If F is implemented a FIFO (First In First Out) queue we have a BFS

The goal check is performed as soon as a node is generated

Informed vs non-informed search

- Besides its own rules, any search algorithm decides where to search next by leveraging some knowledge
- **Non-informed** search uses only knowledge specified at problem-definition time (e.g., goal and start nodes, edge costs), just like we saw in the previous examples
- An **informed** search might go beyond such knowledge
- Idea: using an estimate of how far a given node is from the goal
- Such an estimate is often called a **heuristic**

Estimate of the cost of the optimal path from node v to the goal: $h(v)$

Informed vs non-informed search

- We can enrich DFS and BFS to obtain their an informed versions
- Both search methods break ties in lexicographical order, but it seems reasonable to do that in favor of nodes that are believed to be closer to the goal
- **Hill climbing**
  - A DFS where ties are broken in favor the node with smallest h
- **Beam** (of width w)
  - A BFS where at each level we keep the first w nodes in increasing order of h

Search for the optimal solution

- Now we assume to be interested in the solution with minimum cost (not just any path to the goal, but the cheapest possible)

- To devise an optimal search algorithm we take the moves from BFS. Why it seems reasonable to do that?

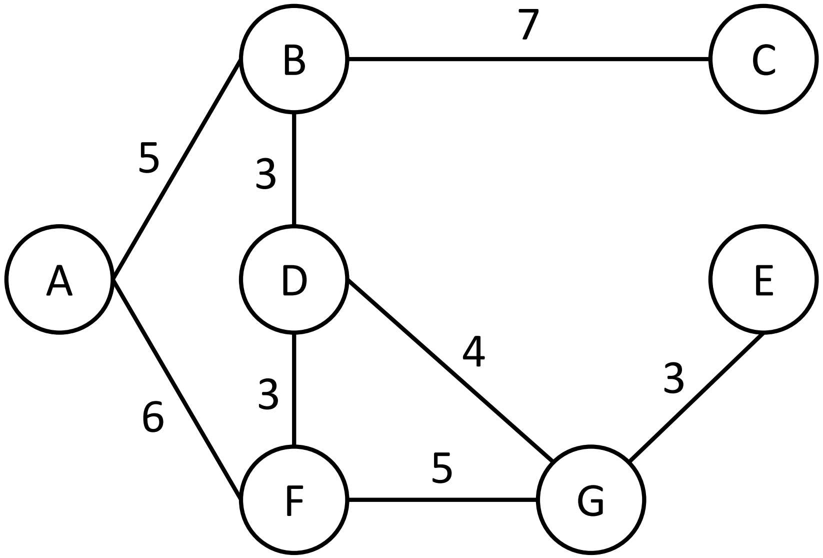
- We generalize the idea of BFS to that of Uniform Cost Search (UCS)

- BFS proceeds by *depth* levels, UCS does that by *cost* levels (as a consequence, if costs are all equal to some constant BFS and UCS coincide)

- Cost accumulated on a path from the start node to v: $g(v)$ (we should include a dependency on the path, but it will always be clear from the context)

- For now let's remove the enqueued list and the goal checking as we know it

Uniform Cost Search (UCS)



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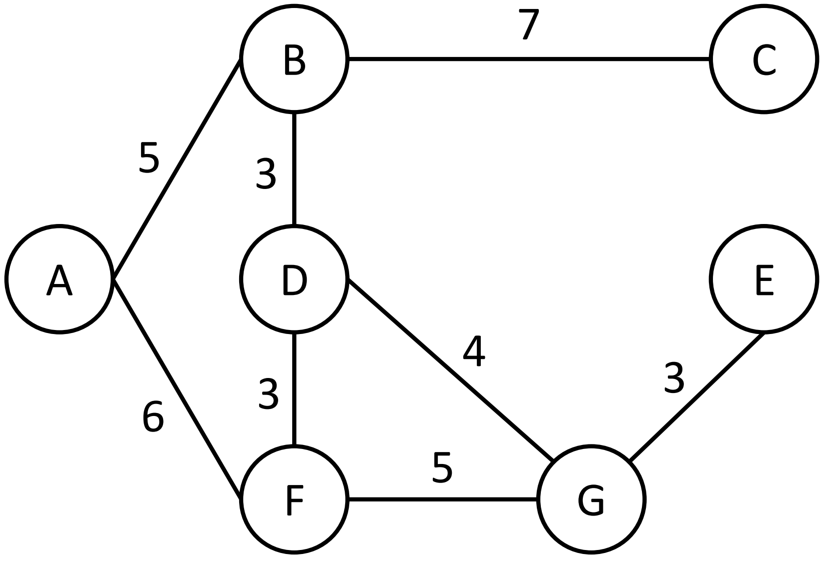

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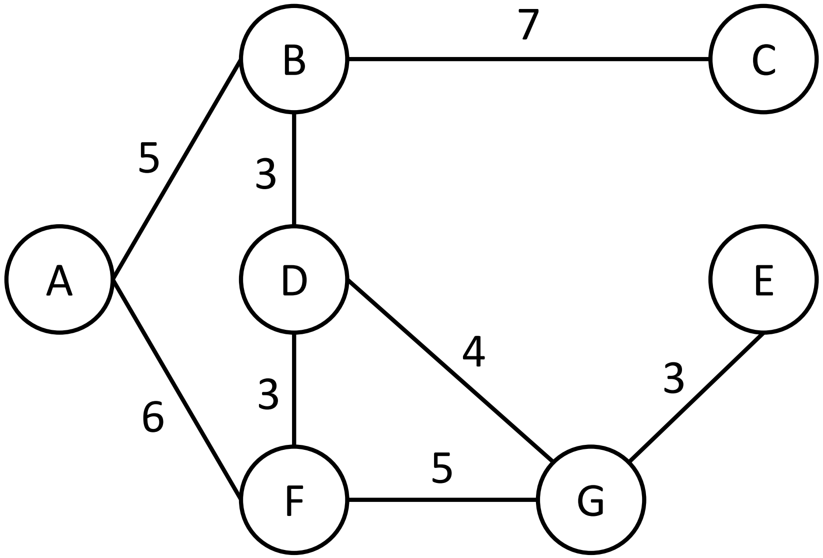

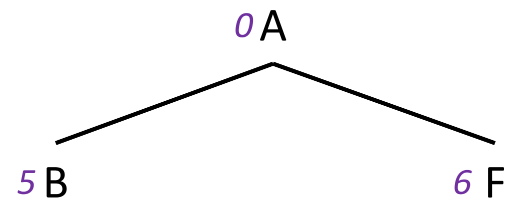

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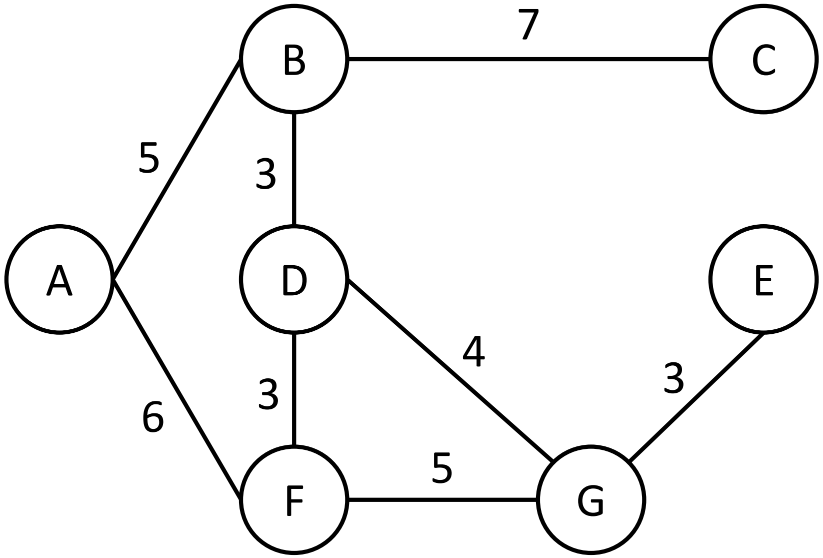

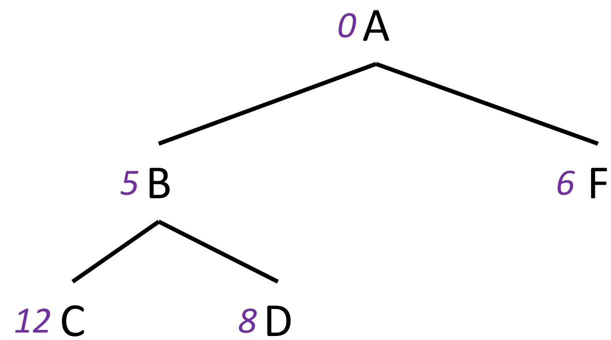

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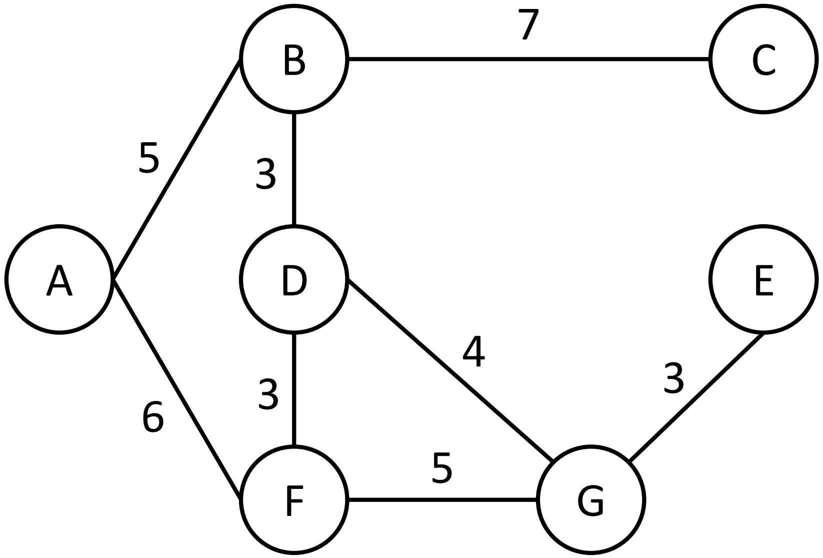

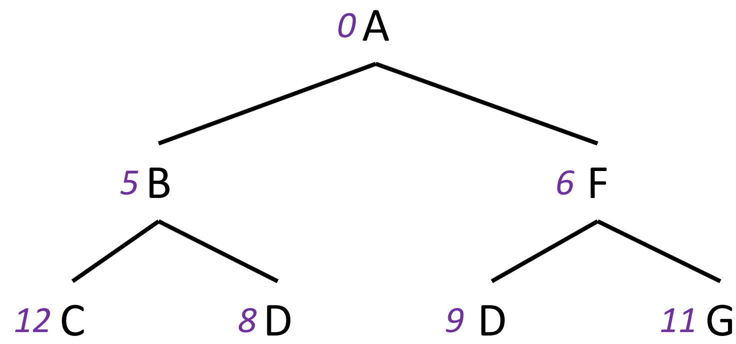

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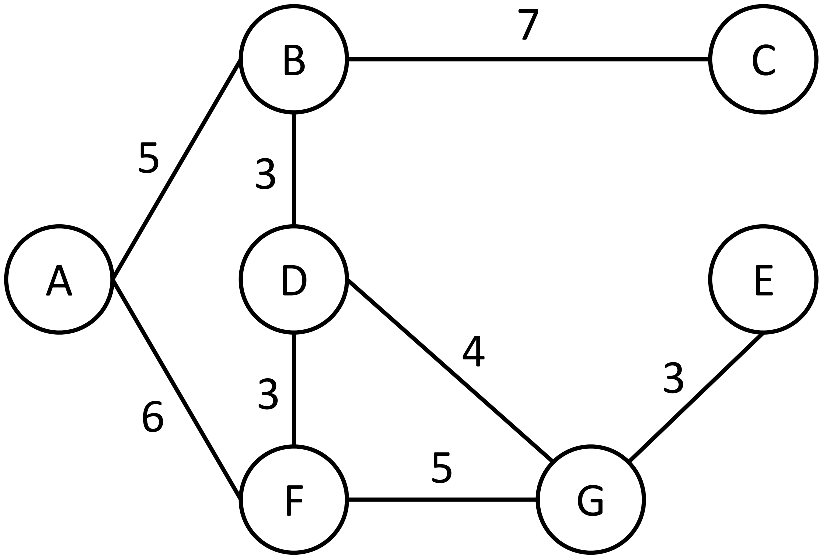

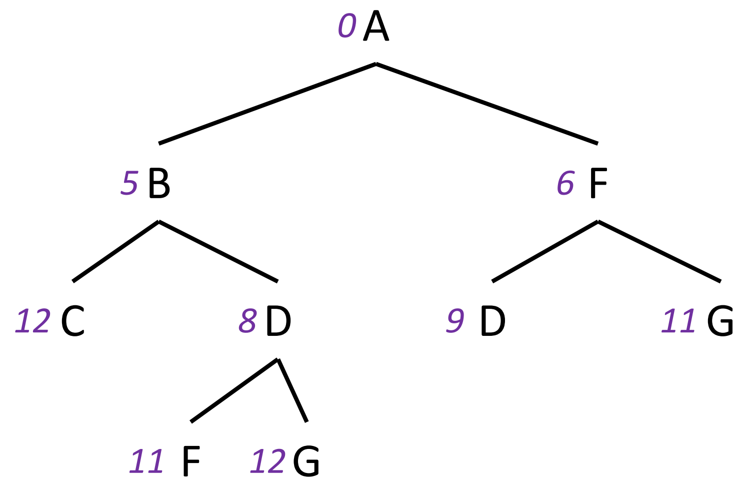

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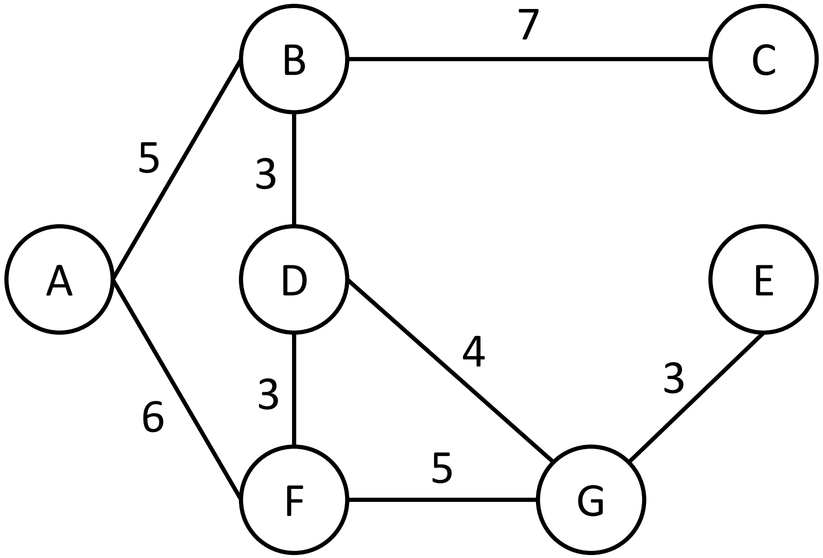

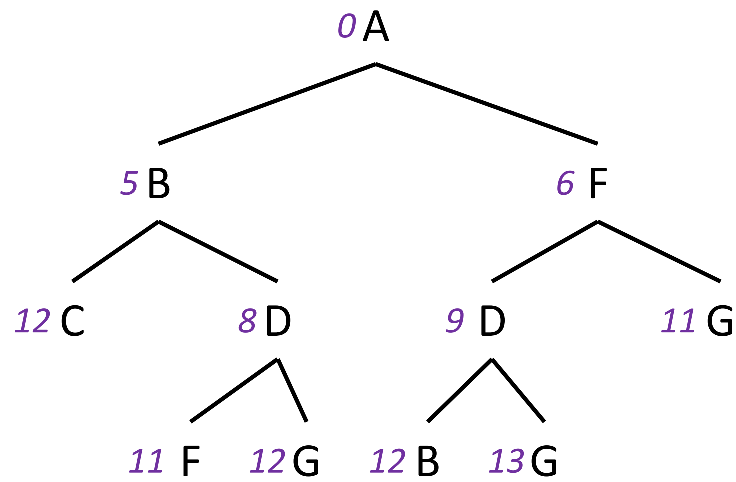

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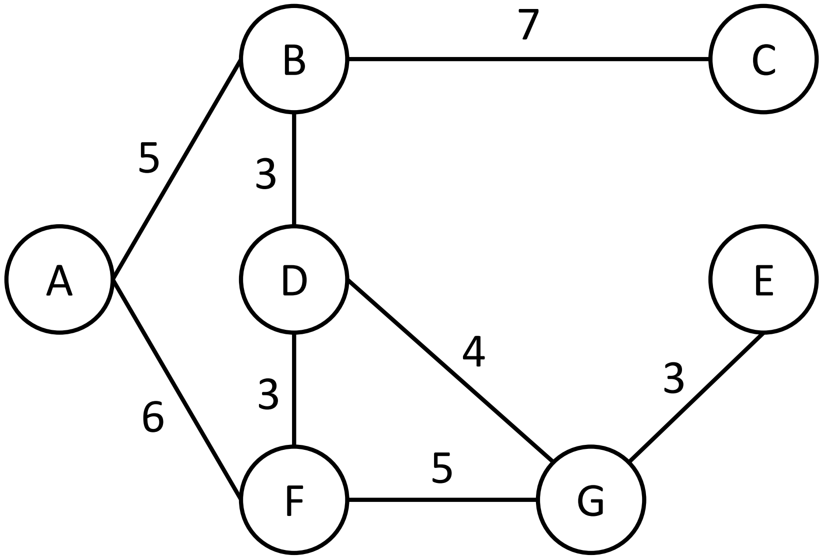

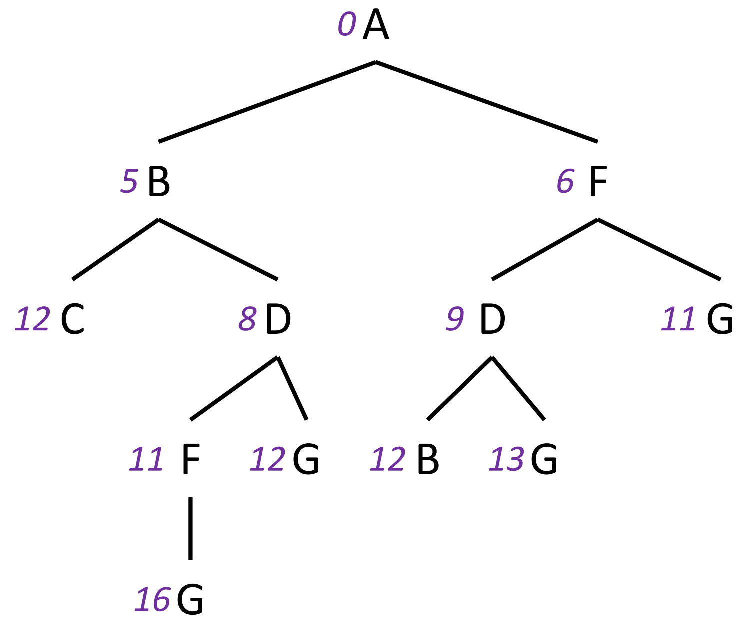

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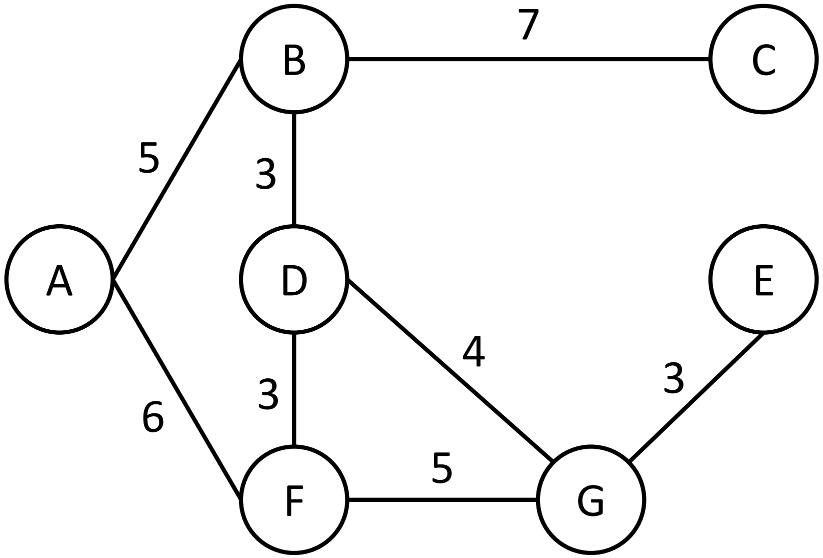

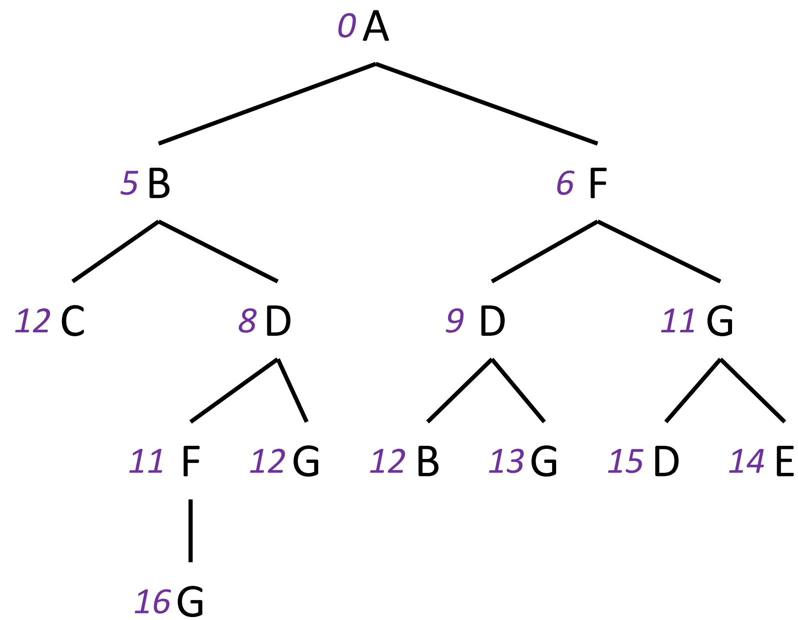

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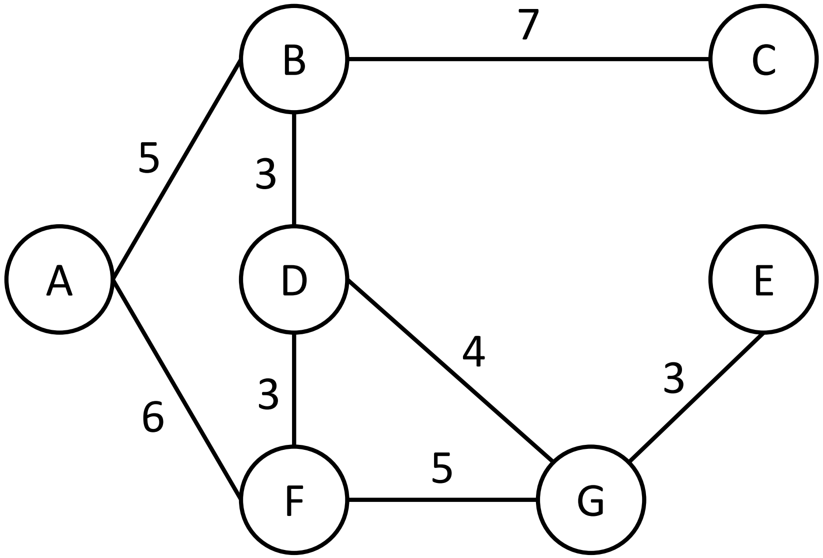

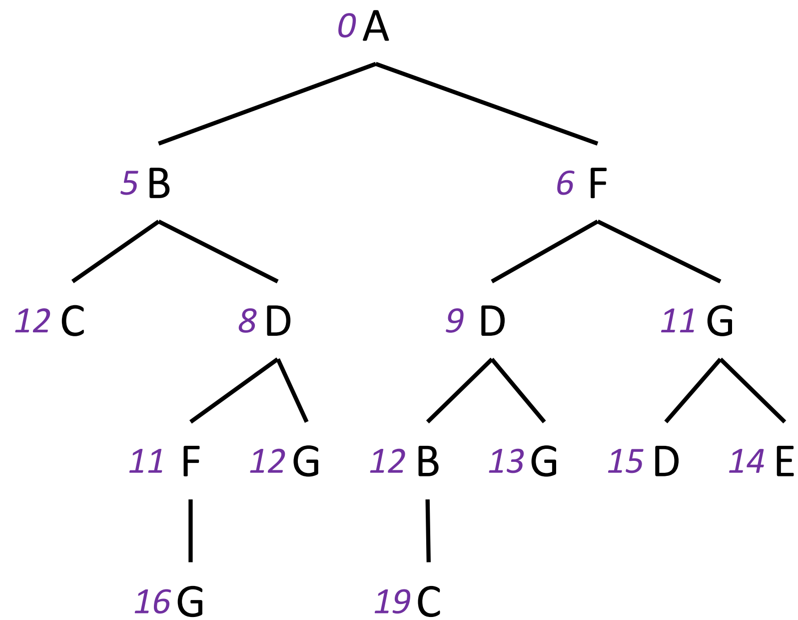

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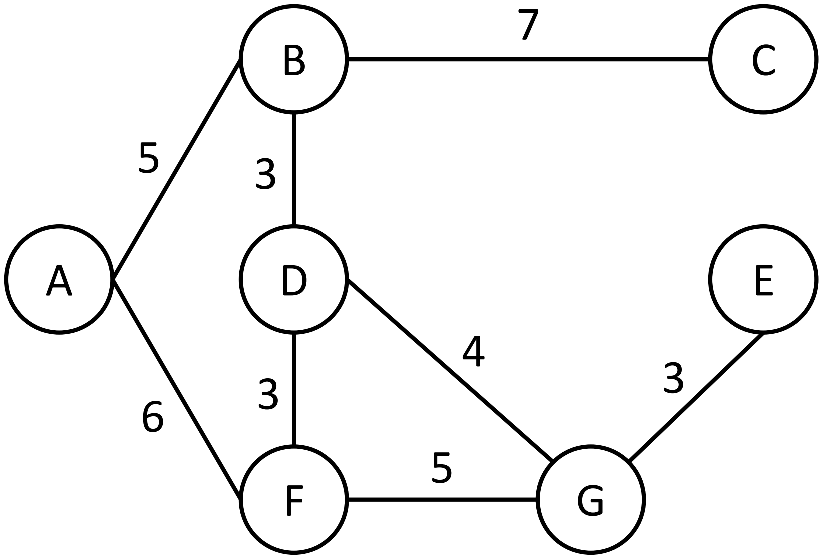
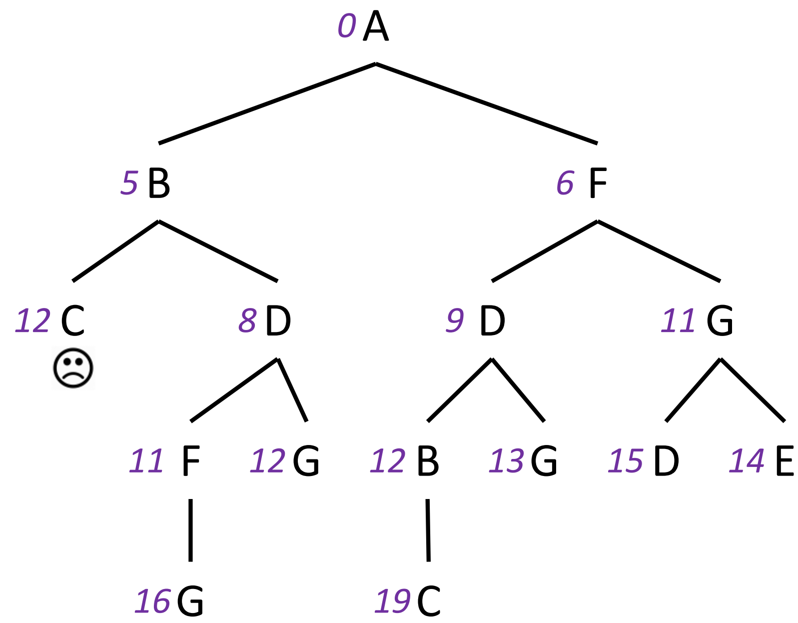
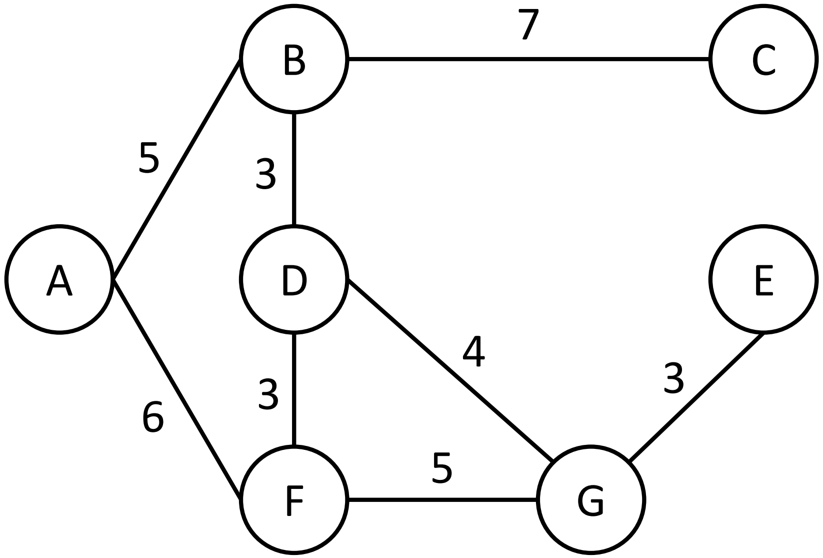
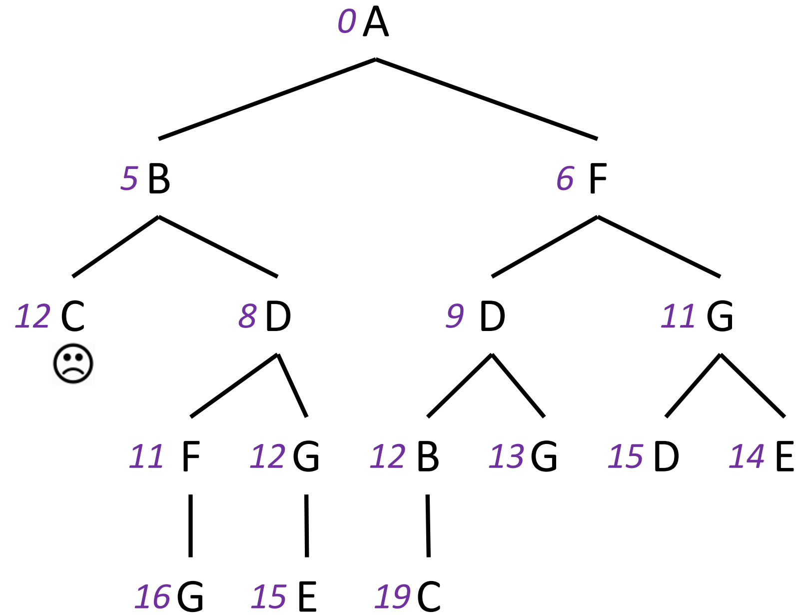




Uniform Cost Search (UCS)





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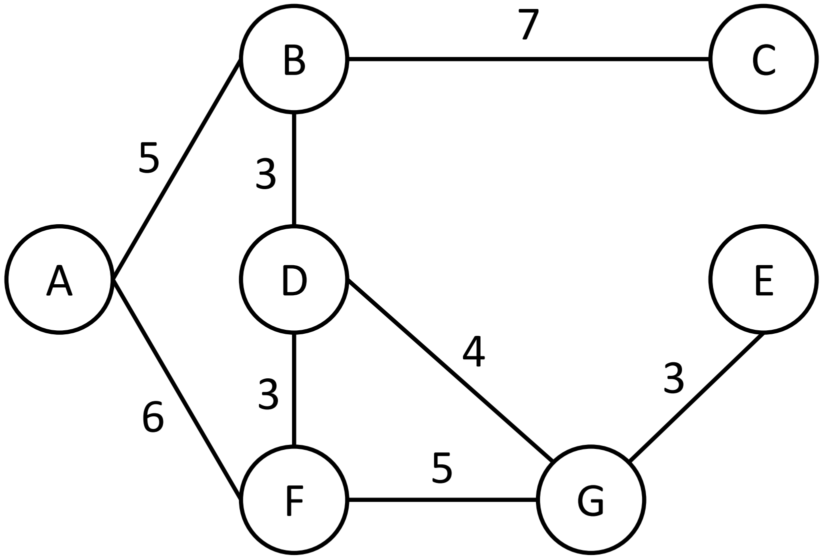

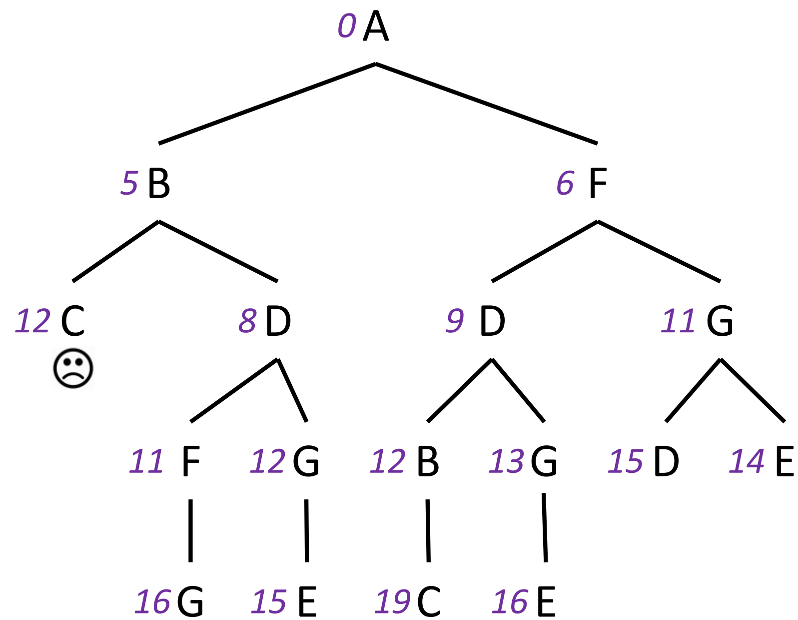

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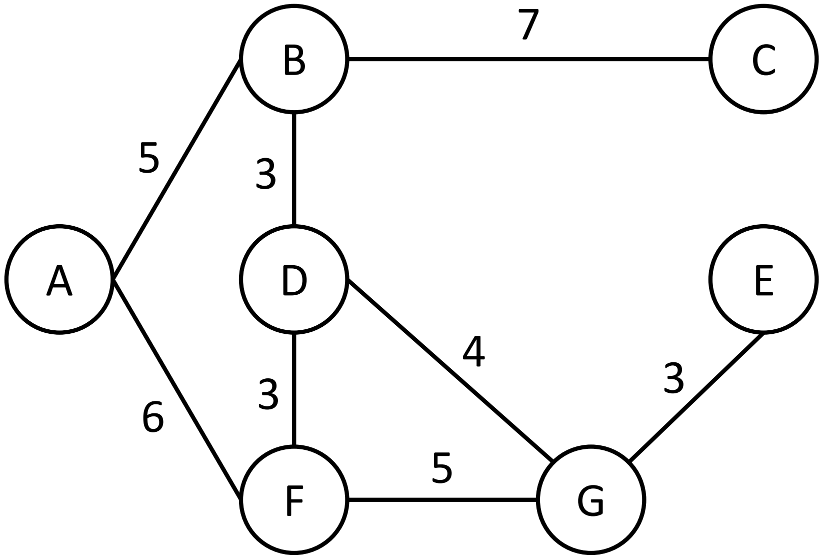

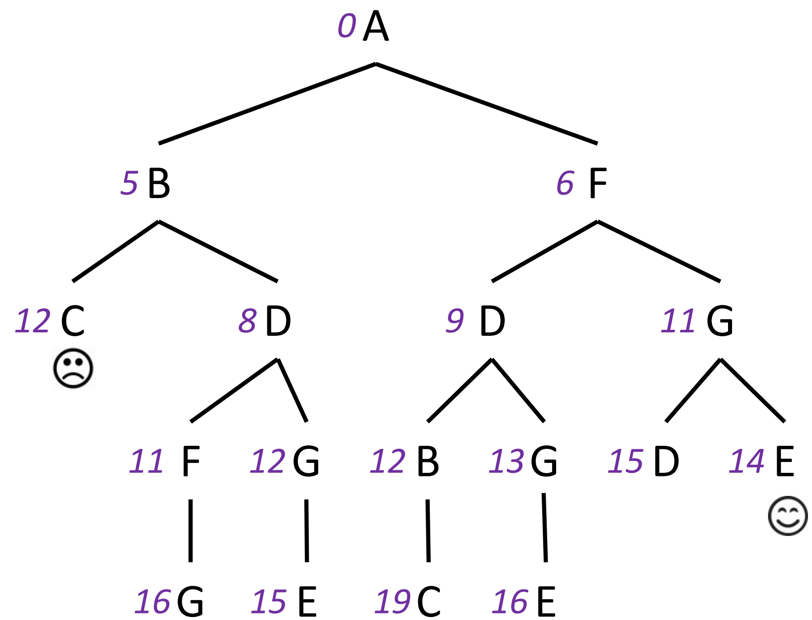

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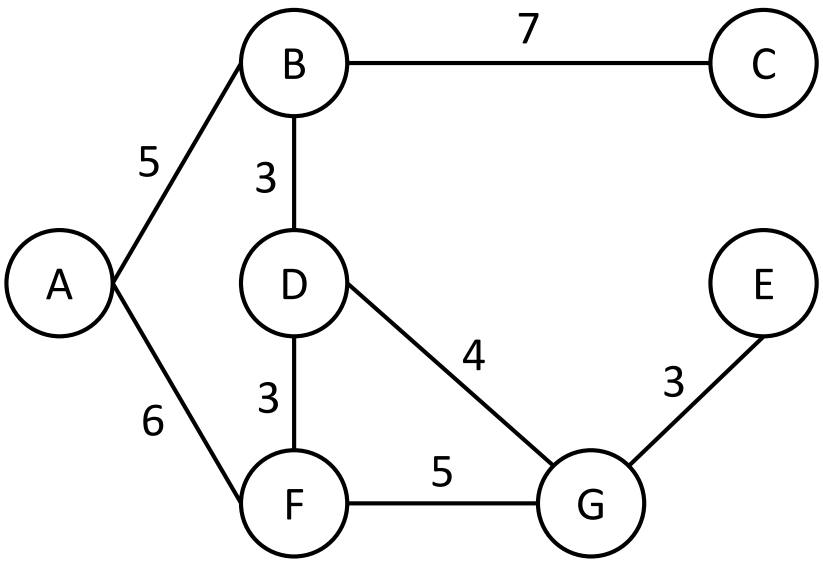
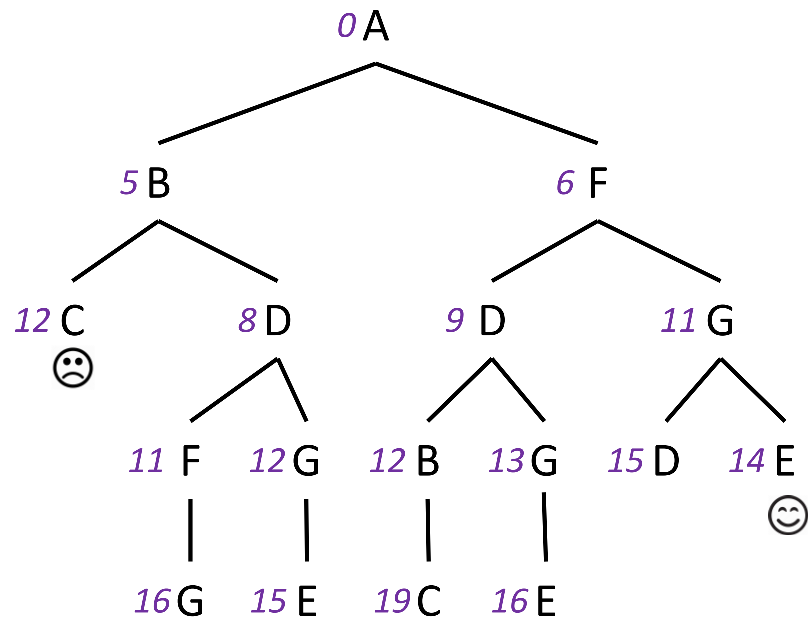


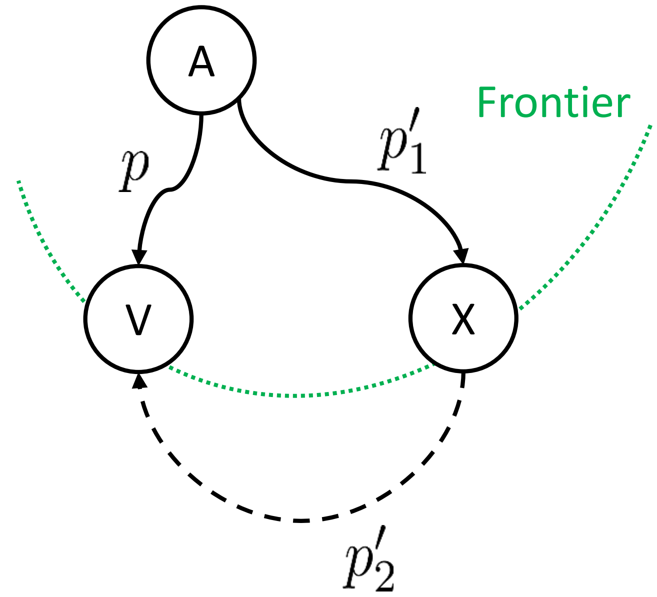


- Have we found the optimal path to the goal? In this problem instance, we can answer *yes* by inspecting the graph
- How about larger instances? Can we prove optimality?
- Actually, we can prove a stronger claim: every time UCS selects **for the first time** a node for expansion, the associated path leading to that node has minimum cost

Optimality of UCS

Hypotheses:

1. UCS selects from the frontier a node V that has been generated through a path p
2. p is not the optimal path to V



Given 2 and the frontier separation property, we know that there must exist a node X on the frontier, generated through a path p'_1 that is on the optimal path $p' \neq p$ to V; let assume $p' = p'_1 + p'_2$

$c(p') = c(p'_1) + c(p'_2) < c(p)$ since, from Hp, p' is optimal

$c(p'_1) < c(p'_1) + c(p'_2) < c(p)$ since costs are positive

$c(p'_1) < c(p)$ X would have been chosen before V, then 1 is false

Optimality of UCS

If when we select for the first time we discover the optimal path, there is no reason to select the same node a second time: **extended list**

Every time we select a node for extension:

- If the node is already in the extended list we discard it
- Otherwise we extend it and we put it the extended list

- (Warning: we are not using an enqueued list, it would actually make the search not sound!)

Sistemi Intelligenti
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Nicola Basilico
Dipartimento di Informatica
Via Celoria 18- 20133 Milano (MI)
Ufficio 4008
nicola.basilico@unimi.it
+39 02.503.16289